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Bayesian Estimation of Quantum States

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1. Motivation

Why?

- Advancement in quantum technology relies greatly on the ability to prepare specific quantum states
- This is done by passing known states through quantum logic gate sequences
- Preparation process is prone to errors due to experimental nature
- We require **a means of testing the accuracy** of the process

What has been done?

- **Quantum State Tomography** – recovers the complete description of a state; resources scale exponentially with system size [1]
- **Trace Distance and Uhlmann Fidelity** – distance metrics which characterise the closeness of states; requires computation of density matrices using quantum state tomography [2]

2. Bayesian Framework

Aim: Investigate whether a Bayesian inference based method can output a probabilistic measure indicative of state fidelity

- **Single qubit pure states** $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle$
- **Pauli measurements** made on states:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow \{|+\rangle, |-\rangle\}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \longrightarrow \{|R\rangle, |L\rangle\}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \{|0\rangle, |1\rangle\}$$

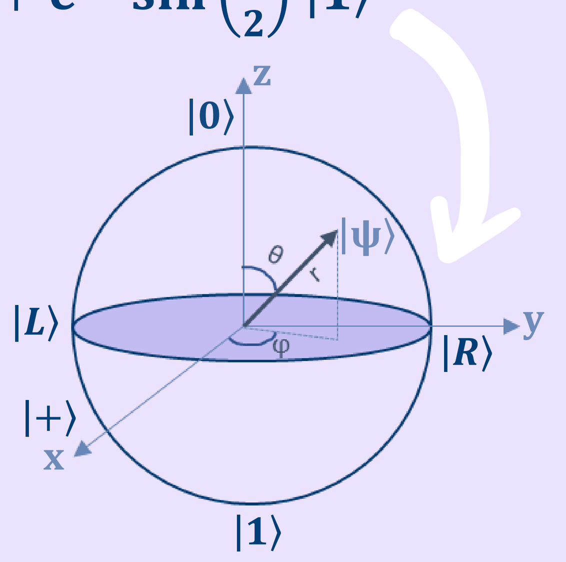


Figure 1: Bloch sphere

- Probabilities of outcomes: **Born's rule**
- For a given dataset, D, the probability that we have some state, ρ, can be found through **Bayesian inference**:

$$P(\rho|D) = NP(D|\rho)P(\rho)$$

Likelihood

$$P(D|\rho) = \prod_i p_i^{n_i}$$

p_i, n_i: Probability and number of outcome i

Uniform Prior

$$P(\rho) = \sin(\theta)$$

(Haar measure)

Normalisation

$$N \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi} \frac{1}{2} d\theta \, P(D|\rho)P(\rho) = 1$$

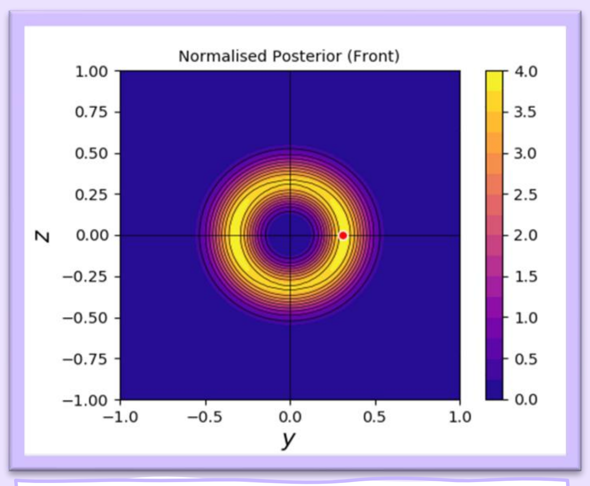


Figure 2: $P(\rho|D)$ for 100x measurements for state $\theta = \frac{\pi}{2}, \varphi = 0.1\pi$

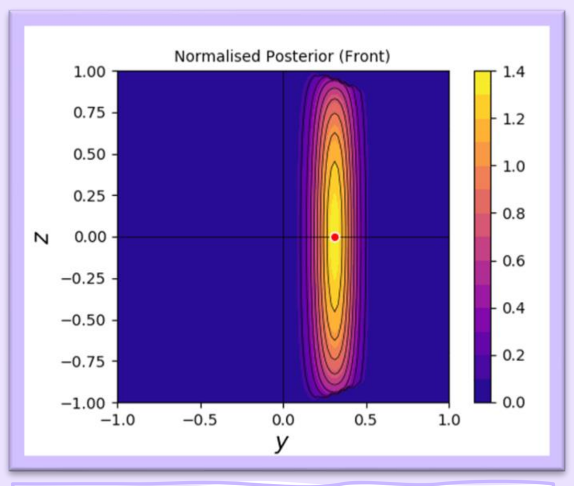


Figure 3: $P(\rho|D)$ for 100y measurements for state $\theta = \frac{\pi}{2}, \varphi = 0.1\pi$

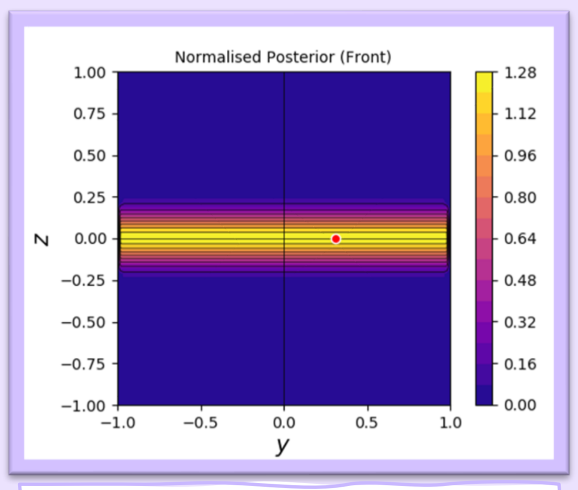


Figure 4: $P(\rho|D)$ for 100z measurements for state $\theta = \frac{\pi}{2}, \varphi = 0.1\pi$

3. Results: Equator States

- **Task**: Distinguish **equator states**, defined by,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle), \qquad \text{where } \theta = \frac{\pi}{2}$$

- **Convenient state space** because:
  - σ_z measurements don't help → reduce relevant Pauli measurements set to $\{\sigma_x, \sigma_y\}$
  - prior is the same for all states

Figure of merit for distinguishing equator states: **Ratio, R**, between probability densities of lab state $|\psi_1\rangle$ and comparison state $|\psi_2\rangle$,

$$R = \frac{P(|\psi_1\rangle)}{P(|\psi_2\rangle)}$$

- For analysis, we
  - define n_i to be the expectation value of outcome i :

$$\langle n_i \rangle = p_i n_I \quad \text{where } I = \{x, y\}$$

  - fix total number of measurements $M = n_x + n_y$

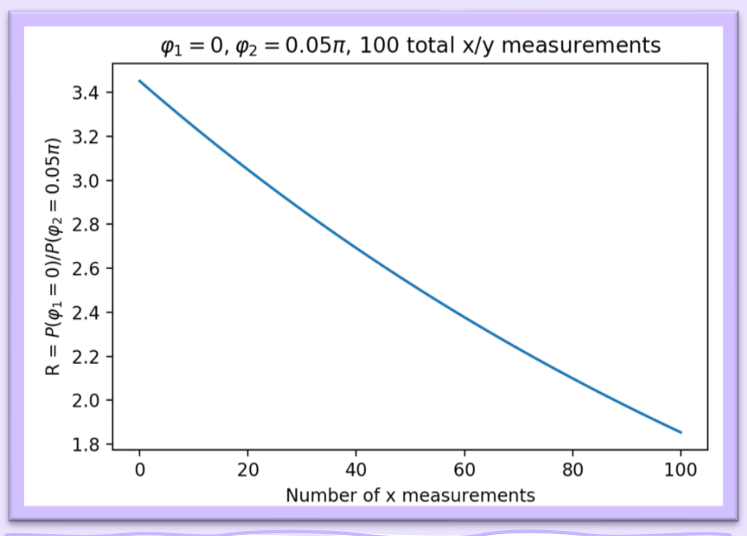


Figure 5: Distinguishability, R, against changing no. x-y measurements for states $\varphi_1 = 0, \varphi_2 = 0.05\pi$

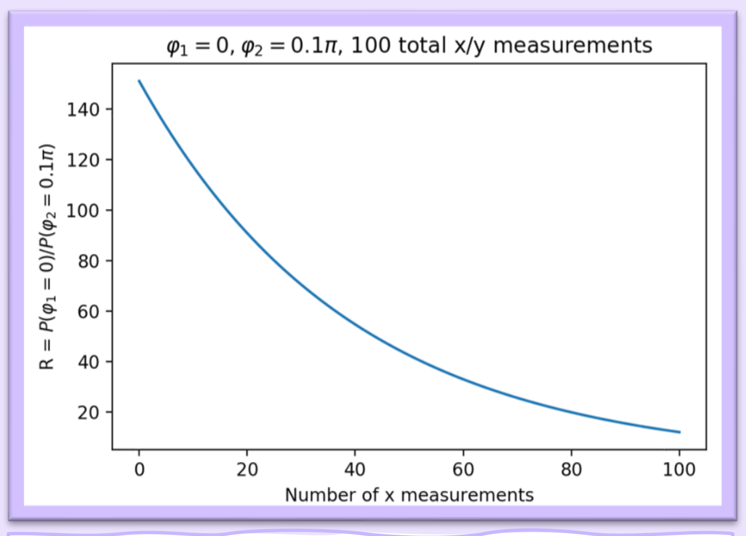


Figure 6: Distinguishability, R, against changing no. x-y measurements for states $\varphi_1 = 0, \varphi_2 = 0.1\pi$

Result:

✔ Maximum y-measurements can better distinguish $|+\rangle$ from close-by states

? Are there states where combining different Pauli measurements give better distinguishability?

4. Further Investigation: Mixed States

Framework has been **generalised to single-qubit mixed states**, i.e., states inside the Bloch sphere

- Add one more variable: r = magnitude of Bloch vector
- New state space: volume of the Bloch sphere

New prior: $r^2 \sin\theta$

New probabilities associated to Pauli measurement outcomes

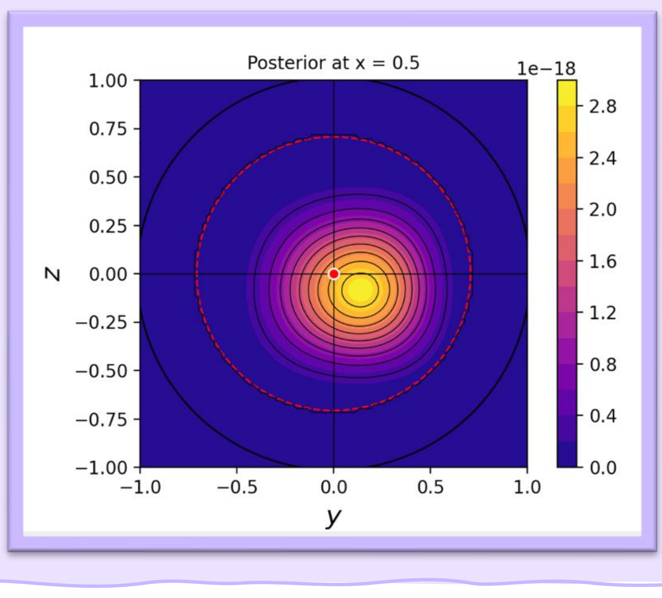


Figure 7: y-z plane projection of $P(\rho|D)$ at x = 0.5 for state $(r, \theta, \varphi) = (0.5, \frac{\pi}{2}, 0)$

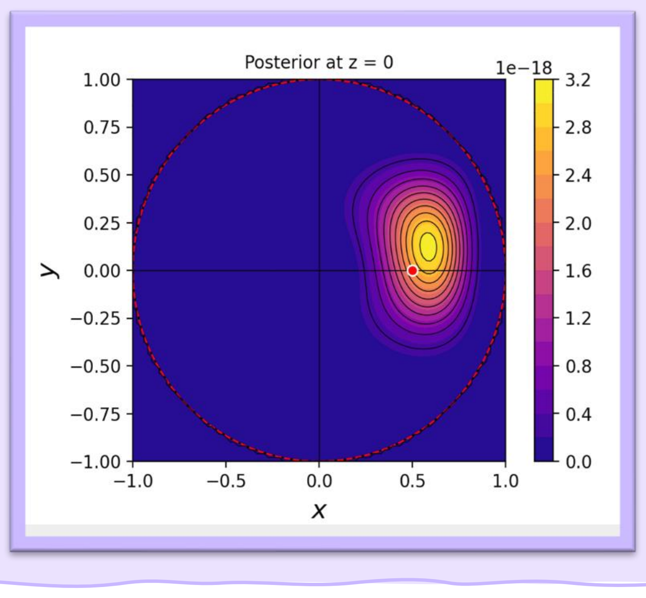
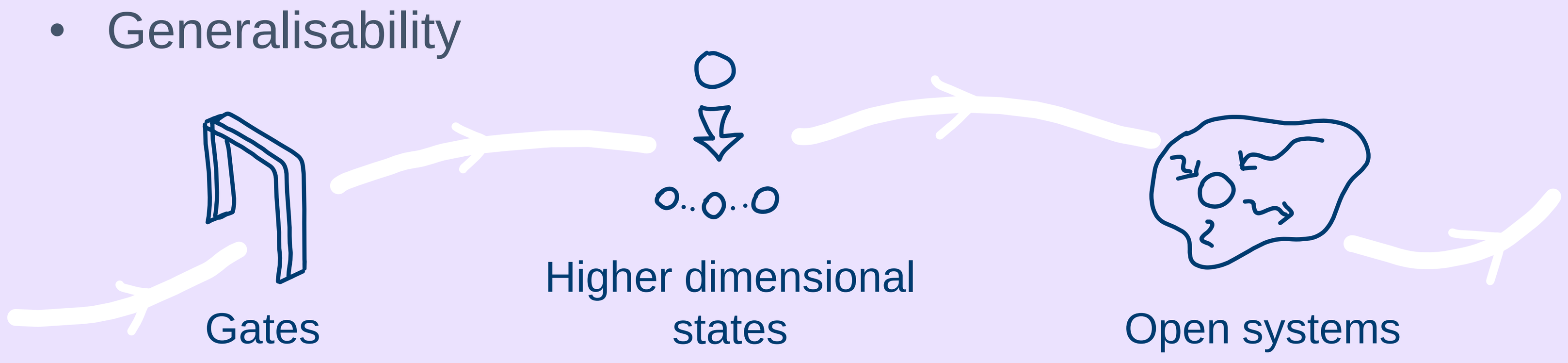


Figure 8: x-y plane projection of $P(\rho|D)$ at z = 0 for state $(r, \theta, \varphi) = (0.5, \frac{\pi}{2}, 0)$

5. Conclusions and Outlook

- We have built a framework which takes in data from different Pauli measurements of a single-qubit quantum state and outputs a probability density distribution which localises the state
- We are investigating whether combining measurements from different Pauli observables improve state discrimination analytically and numerically (with pseudo data)

- Interesting future paths to explore:
  - Generalisability



  - Test framework against current conventional estimation methods
  - How the framework will perform under adaptive measurements

References

[1] R. Blume-Kohout, "Optimal, reliable estimation of quantum states", New Journal of Physics, vol. 12, no. 4, p. 043 034, Apr. 2010. doi: 10.1088/1367-2630/12/4/043034.

[2] R. Chen, Z. Song, X. Zhao, and X. Wang, "Variational quantum algorithms for trace distance and fidelity estimation", Quantum Science and Technology, vol. 7, no. 1, p. 015019, Dec. 2021. doi: 10.1088/2058 - 9565/ac38ba. 11