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Positivity Bounds for Scalar Field Theories

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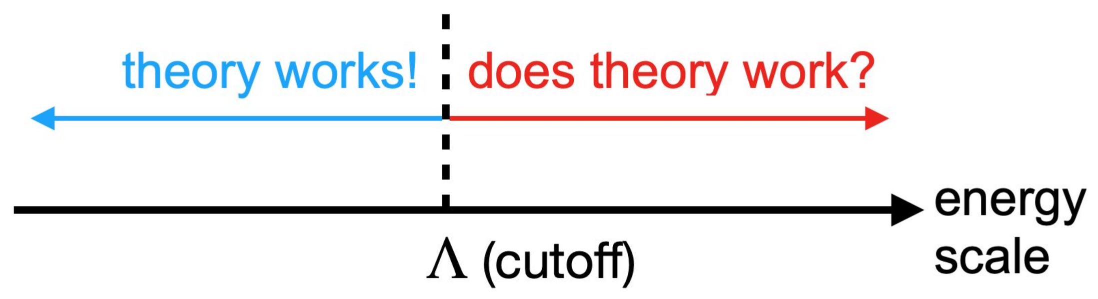
Background & Motivation

S-Matrix:
In Quantum Field Theory, the S-matrix (scattering matrix) is an important object used to calculate the scattering amplitude for possible interactions.

$$\hat{S} = \hat{\mathcal{T}} \exp\Big(-i \int_{-\infty}^{+\infty} \hat{H}_{\mathrm{I}}(t')\, dt' \Big)$$

Low Energy Effective Field Theory (LEEFT) :
Describes the low energy part of the physics without getting tied up to the high energy regime.



- **If works:** local, causal, unitary and Lorentz invariant — "UV complete".
- **If does not work:** non-local. Possible manifestations: superluminal signal propagation, violation of analyticity constraints, etc.

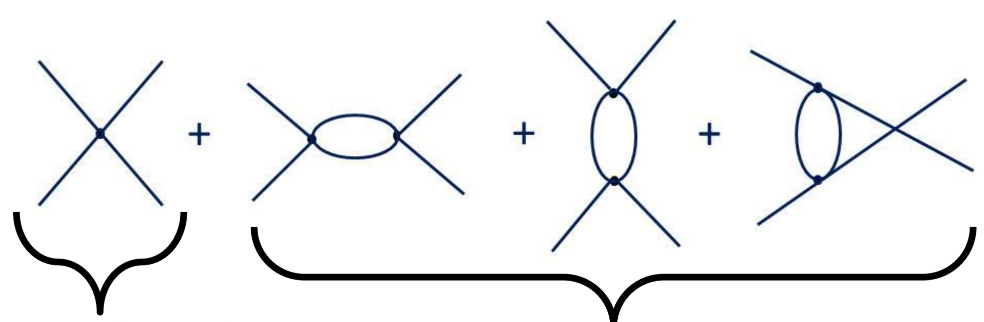
Q: How to determine?
A: Positivity bounds! These place constraints on the LEEFTs. Violating any of them directly implies the absence of any possible well-defined UV completion for the theory.

Our Work

1. Massive scalar field ϕ, calculate 2-2 scattering amplitude, A(s,t), for:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \frac{g}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^2$$

to one-loop order (low energy regime).



2. Regulate the resulting loop momentum integrals using **dimensional regularisation** and **renormalisation,** since they are divergent in 4 spacetime dimensions (d = 4).

Dimensional regularisation: Introduce to the coupling an arbitrary mass parameter, μ, and work in d = 4 - ε dimensions to force the integrals to converge. This isolates the divergences in the form of local 1/ε terms.

Renormalisation: Absorb the divergences into various couplings whilst keeping the physical (and hence measurable) parameters finite and independent of μ.

3. Apply Cauchy's integral formula in the complex s plane at fixed t.

$$\mathcal{A}(s,t) = \frac{1}{2\pi i}\oint_C \frac{\mathcal{A}(s',t)}{s'-s}\,\mathrm{d}s'$$

(A(s,t) = A(u,t) due to crossing symmetry)

Using the contour shown, we get contributions from the two physical poles, two branch cuts, and two subtraction functions, a(t) & b(t).

$$\mathcal{A}(s,t) = a(t) + b(t)s + \frac{\mathrm{Res}(s=m^2,t)}{m^2-s} + \frac{\mathrm{Res}(s=3m^2-t,t)}{3m^2-s-t} + \int_{4m^2}^{\infty}\left[\frac{(s-\mu_{\mathrm{p}})^2\mathrm{Im}\mathcal{A}(\mu,t)}{(\mu-\mu_{\mathrm{p}})^2(\mu-s)} + \frac{(u-\mu_{\mathrm{p}})^2\mathrm{Im}\mathcal{A}(\mu,t)}{(\mu-\mu_{\mathrm{p}})^2(\mu-u)}\right]\frac{\mathrm{d}\mu}{\pi}$$

The latter arise from introducing an arbitrary double pole at subtraction point μ_p, ensuring A(s',t) converges in the limit |s'| → ∞ (since |A(s,t)| < s^2).

5. Find the imaginary part of the amplitude, Im[A(s,t)], resulting from contour integration around the two branch cuts.

6. Subtract physical poles (s = m^2, u = m^2) and differentiate twice to remove subtraction functions (taking μ_p = 0). **Leading positivity bound:** $\partial_s^2\, \mathcal{A}'(s,t)\big|_{s=t=0} > 0$

7. Since we trust LEEFT up to some energy cut-off Λ* (Λ* > $4m^2$), a stronger argument can be made by shifting the branch cut:

$$\begin{aligned}\partial_s^2\, \tilde{\mathcal{A}}'(s,t)\big|_{s=t=0} &= \partial_s^2\, \mathcal{A}'(s,t)\big|_{s=t=0} - \frac{2}{\pi}\int_{4m^2}^{\Lambda^{*2}} d\mu \frac{\mathrm{Im}\mathcal{A}(\mu,0)}{\mu^3} - \frac{2}{\pi}\int_{4m^2}^{\Lambda^{*2}} d\mu \frac{\mathrm{Im}\mathcal{A}(\mu,0)}{(\mu-4m^2)^3} \\ &= \frac{2}{\pi}\int_{\Lambda^{*2}}^{\infty} d\mu \frac{\mathrm{Im}\mathcal{A}(\mu,0)}{\mu^3} + \frac{2}{\pi}\int_{\Lambda^{*2}}^{\infty} d\mu \frac{\mathrm{Im}\mathcal{A}(\mu,0)}{(\mu-4m^2)^3} \\ &> 0\end{aligned}$$

Results & Conclusion

- Leading positivity bound: $\partial_s^2\, \mathcal{A}'(s,t)\big|_{s=t=0} = \frac{g}{2\Lambda^4} + \frac{3g^2m^4}{64\pi^2\Lambda^8}\left[\ln\left(\frac{\mu^2}{m^2}\right)\right] > 0$
  As expected, the loop contribution vanishes when μ = m. Consequently: $g > 0$
- Leading improved positivity bound:

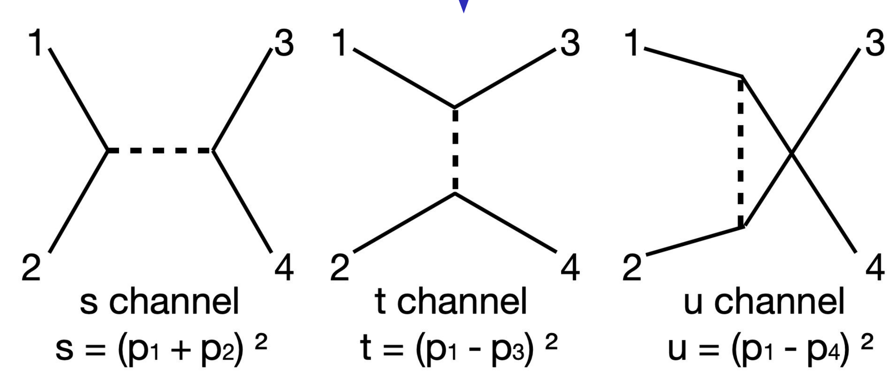
$$\begin{aligned}\partial_s^2\, \tilde{\mathcal{A}}'(s,t)\big|_{s=t=0} = &\frac{g}{2\Lambda^4} + \frac{3g^2m^4}{64\pi^2\Lambda^8}\left[\ln\left(\frac{\mu^2}{m^2}\right)\right] - \frac{g}{10240\pi\Lambda^8}\Bigg[\sqrt{1-\frac{4m^2}{\Lambda^{*2}}}\Big(\frac{3}{2}g\Lambda^{*8} - \frac{64}{3}g\Lambda^{*6}m^2 + \frac{412}{3}g\Lambda^{*4}m^4 - 624g\Lambda^{*2}m^6 - \\ &\frac{4232}{5}gm^8 + \frac{256gm^{10}}{5\Lambda^{*2}} + \frac{128gm^{12}}{5\Lambda^{*4}}\Big) + 320\Lambda^{*4}\pi\Lambda^4 - 5120\Lambda^{*2}m^2\pi\Lambda^4 + \frac{20480m^6\pi\Lambda^4}{\Lambda^{*2}} - \frac{5120m^8\pi\Lambda^4}{\Lambda^{*4}} - \\ &96(11gm^8 + 160m^4\pi\Lambda^4)\ln\Big(1-\sqrt{1-\frac{4m^2}{\Lambda^{*2}}}\Big) + 96(11gm^8 - 160m^4\pi\Lambda^4)\ln\Big(1+\sqrt{1-\frac{4m^2}{\Lambda^{*2}}}\Big)\Bigg] \\ &> 0\end{aligned}$$

- Interaction coupling g must satisfy this bound in order to have a well-defined UV completion

Theory

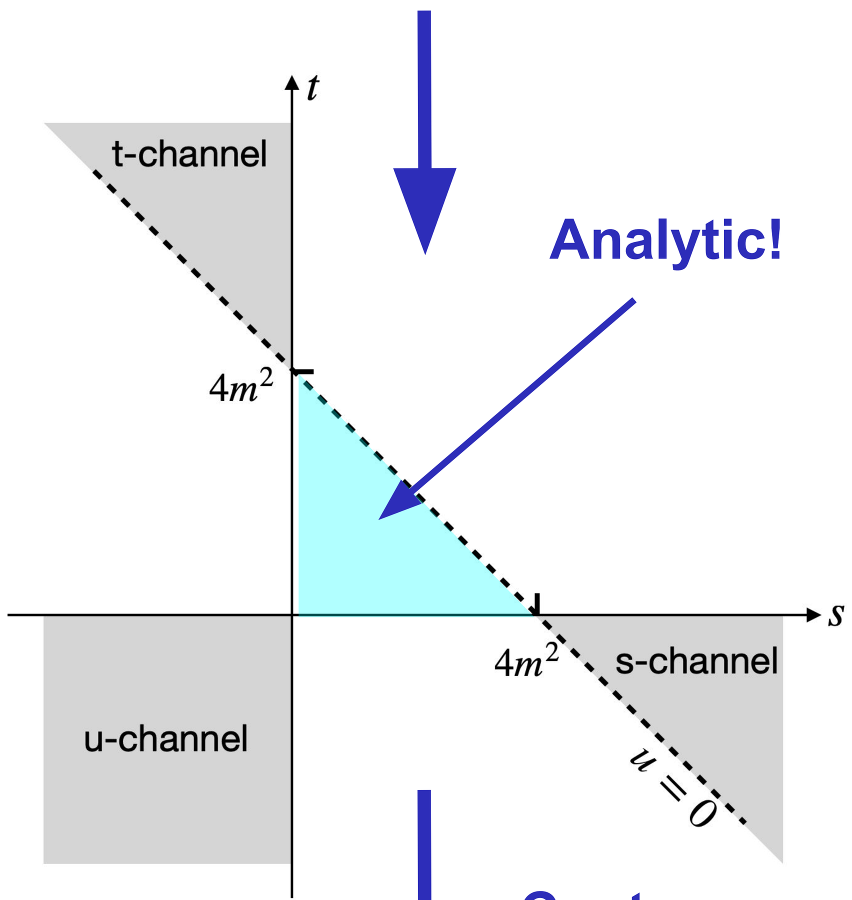
Unitarity + Locality + Causality
(Analyticity constraints on S-matrix)

Crossing Symmetry

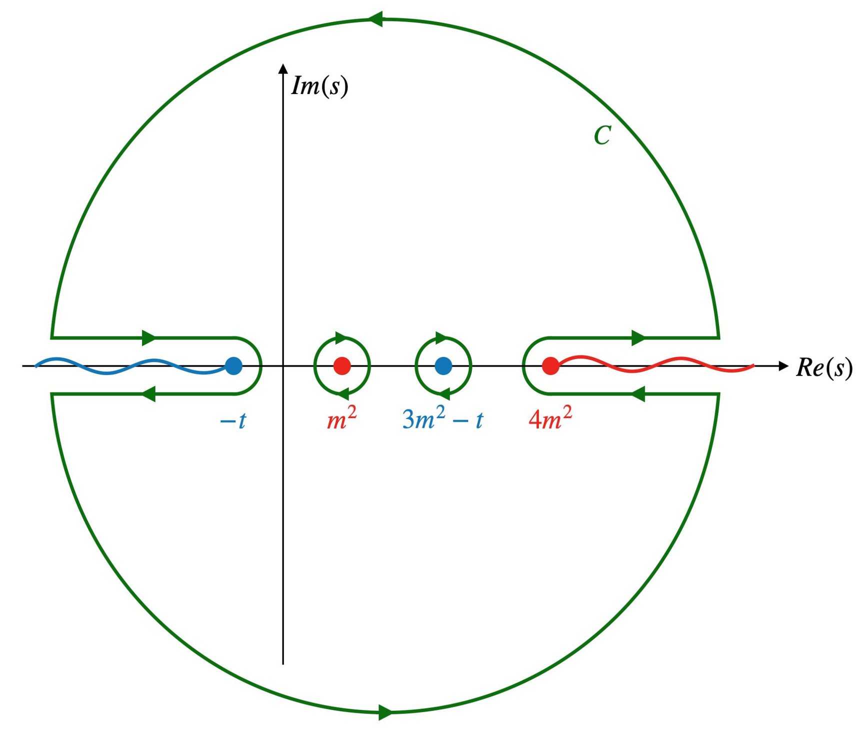


$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$

Analytic!



Contour Integral



$$\mathcal{A}(s,t) \propto \int_{4m^2}^{\infty} \mathrm{d}\mu\left[\frac{\mathrm{Im}\mathcal{A}(\mu,t)}{\mu-s} + \frac{\mathrm{Im}\mathcal{A}(\mu,t)}{\mu-u}\right]$$

$$\partial_s^N\partial_t^M \int_{4m^2}^{\infty}\frac{\mathrm{d}\mu}{\pi}\left[\frac{s^2\mathrm{Im}\mathcal{A}(\mu,t)}{\mu^2(\mu-s)} + \frac{u^2\mathrm{Im}\mathcal{A}(\mu,t)}{\mu^2(\mu-u)}\right]\Bigg|_{s=t=0} > 0 \quad \text{for} \quad N \geq 1,\ M \geq 0$$

—— Positivity bounds !

Future Directions

- Impose full crossing symmetry to derive sets of nonlinear bounds
- Derive positivity bounds for theories that violate Lorentz invariance
- Extend into massless limit may provide implications on the theories coupled to gravity