Imperial College London

Spectral Geometry of Causal Sets

Julia Dannemann-Freitag & Antoine Vauterin

Theoretical Physics Group
Supervisor: Dr. Yasaman K. Yazdi

Motivation

Quantum Gravity is a difficult problem in part because Quantum Field Theory uses the mathematics of analysis while General Relativity is formulated in terms of differential geometry. An area of mathematics that can bridge the gap between the analytical and the geometrical is Spectral Geometry, which is the study of geometry through the spectra of operators. Traditionally, Spectral Geometry focuses on Riemannian manifolds, but spacetime is usually described as a Lorentzian manifold. Causal Set Theory is a natural framework for extending Spectral Geometry to spacetimes, since it is frame independent and deals with discrete spacetimes, with operators that are finite matrices.

Background

Causal Set Theory

Causal Set Theory (CST) is an approach to Quantum Gravity in which spacetime is fundamentally discrete. A causal set consists of spacetime points called elements and their causal relations, described by a partial order which determines whether an element causally precedes another.

A causal set (or causet) $\mathcal{C}$ is a set with a partial order relation $\preceq$ obeying [1]:

- Reflexivity: for all $x \in \mathcal{C}$, $x \preceq x$
- Transitivity: for all $x, y, z \in \mathcal{C}$, $x \preceq y \preceq z \Rightarrow x \preceq z$
- Antisymmetry: for all $x, y \in \mathcal{C}$, if $x \preceq y$ and $y \preceq x \Rightarrow x = y$
- Local finiteness: for all $x, y \in \mathcal{C}$, $|I(x, y)|$ is finite, where $I(x, y) := \{z \in \mathcal{C} \text{ with } x \preceq z \preceq y\}$

A causet is said to be faithfully embedable in a spacetime manifold (thus making it "manifold-like") if the causal relations between elements correspond to those of points randomly distributed in that manifold according to a Poisson distribution. Thus, if we wish to study a specific manifold through CST, we can sprinkle into that manifold by distributing the points in it randomly and calculating the causal relations using its metric.

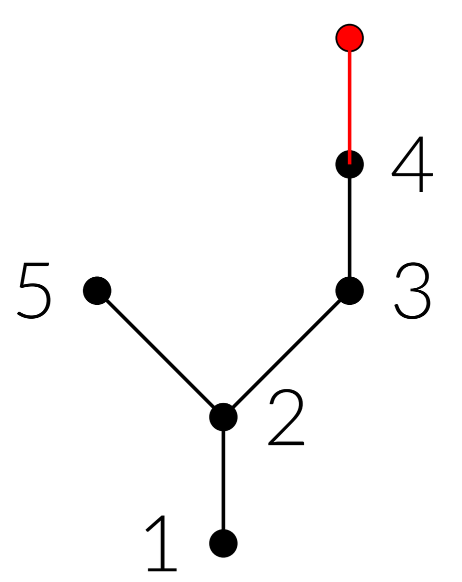

Figure 1. The Hasse diagram of a 5-element causet (in black). The edges connect elements that immediately precede each other. In red is a quill perturbation added to the original causet.

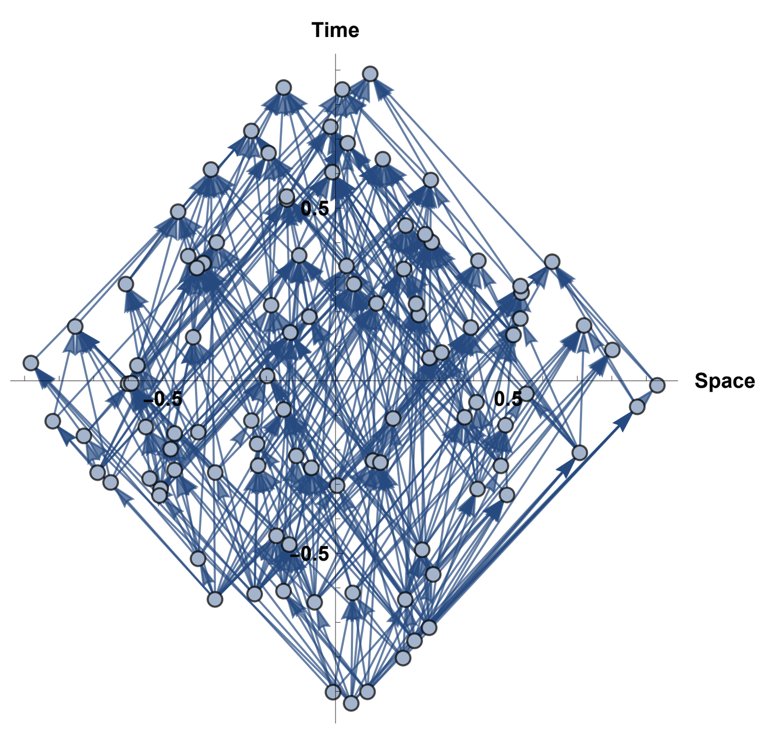

Figure 2. A 100-element causet sprinkled into a flat 2D causal diamond. The speed of light is taken to be unity and arrows indicate the direction of causality.

Spectral geometry

Spectral geometry is the the branch of mathematics that studies the spectra of canonical operators. Most famous is the question "Can one hear the shape of a drum?" [3], or in other words, can a drum's boundary be determined from the resonances of its surface? Another important result is Weyl's law, which states that the eigenvalues of the Laplacian on a Riemannian manifold follow a power law.

We are interested in applying these techniques to Lorentzian manifolds, for which CST offers a discrete framework. Since causets are discrete, so are their operators and thus their spectra, making them easier to study as they will have a finite number of eigenvalues.

Spectral geometry might be of use in determining whether a causet is manifold-like or not. It could also allow us to determine the dimension or curvature of a causet.

Methodology

Types of Causal Sets

We focus our study on specific causets. In particular, we want to study a variety of properties (dimension, curvature...). The different causet types we consider are:

- Causet embedded in flat spacetime
- Causal diamonds: the intersection of the past light-cone of an event with the future light-cone of another event, here taken in flat spacetime. An example of a 2d causal diamond is shown in figure 2
- Causet embedded in de Sitter spacetime: de Sitter spacetime is a curved spacetime with constant positive curvature
- KR order: the entropically dominant class of non-manifold-like causal sets [4]
- Chain: a totally ordered set

Operators

We also had to choose which operators we would study, with the obvious choice being the two central operators of CST: the adjacency matrices. The causal matrix encodes the causal relations while the link matrix encodes what elements immediately precede ($\prec *$) each other. Both fully specify a causet and are defined as:

$$C_{xy} = \begin{cases} 1, & \text{for } x \prec y \\ 0, & \text{otherwise} \end{cases} \qquad L_{xy} = \begin{cases} 1, & \text{for } x \prec *y \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

These operators can be expressed as upper triangular matrices and thus have a trivial spectrum. However, we can take combinations of these matrices to build operators with non-trivial spectra. We can take the self-adjoint and anti-self-adjoint parts: $M_{SA} = \frac{1}{2}(M + M^\dagger)$ and $M_{ASA} = \frac{1}{2}(M - M^\dagger)$. We can also take their commutator: $[M_{SA}, M_{ASA}]$.

Connections to Spectral Dimension

The spectral dimension is a measure of dimension first developed for fractals, defined in terms of the return probability, within a return time, of a random walker. This was adapted to CST by treating the causet as an undirected graph, with each move along an edge taking unit time [2]. Since the adjacency matrix of the undirected version of the causet is $L + L^T$, the SA part of the link matrix, we can, through approximation, relate the eigenvalues of this to the spectral dimension:

$$d_S(\sigma) = -2\frac{\partial \ln(P_r(\sigma))}{\partial \ln(\sigma)} \approx -2\sigma\left(\sigma n_{avg} + \frac{1}{\sum_i \lambda_i}\frac{\partial \sum_i \lambda_i}{\partial \sigma}\right) \tag{2}$$

where σ is the return time and $P_r(\sigma)$ is the return probability for time σ averaged across the causet, n_{avg} is the average number of edges coming from each vertex and the λ_i are the eigenvalues of the SA part of the link matrix to the power of σ.

Results and Outlook

Spectra of Causets

We examine spectra of causal sets directly, using the SA part of the link matrix. The spectra clearly obey a power law until a certain point. We can see that this power law depends on the dimension of the causal set and is much flatter for KR sets. It is also dependent on the size of the causet, but this dependence is diminishing with size. Thus, it may be possible to define a lower bound on point density above which the spectrum can distinguish a given dimension of a causet. Preliminary results of this are shown in figure 3.

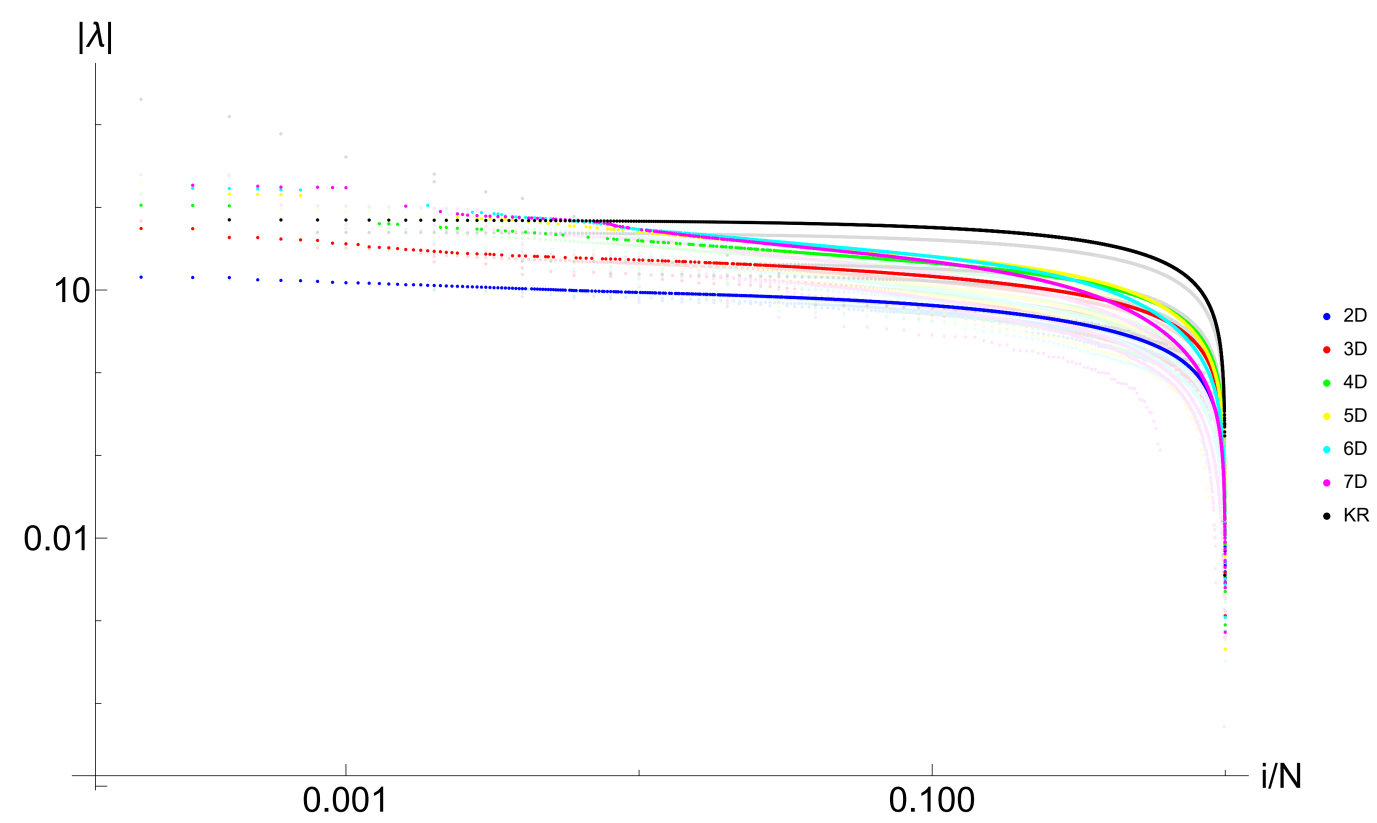

Figure 3. Log-log plot of spectra of two times the self-adjoint part of the link matrix of causal sets sprinkled into flat space in 2,3,4,5,6 and 7 dimensions and KR sets. For each type there are causets of sizes N = 100, 500, 1000, 5000 and 10 000 (shown in bolder colours). The plot shows the magnitude of the spectra vs. the index of each eigenvalue divided by the size of the causet.

Quill perturbations

A quill perturbation is the systematic addition of elements and links to a causal set [5]. For example, in figure 1 we have performed a quill perturbation by adding a single element and link (in red). We are interested in how the spectra of different causets behave under quill perturbations. A way to compare two spectra is the spectral distance:

$$\sigma(\mathcal{C}_1, \mathcal{C}_2) = \sum_{i=1}^{N} |\lambda_i^{(2)} - \lambda_i^{(1)}| \tag{3}$$

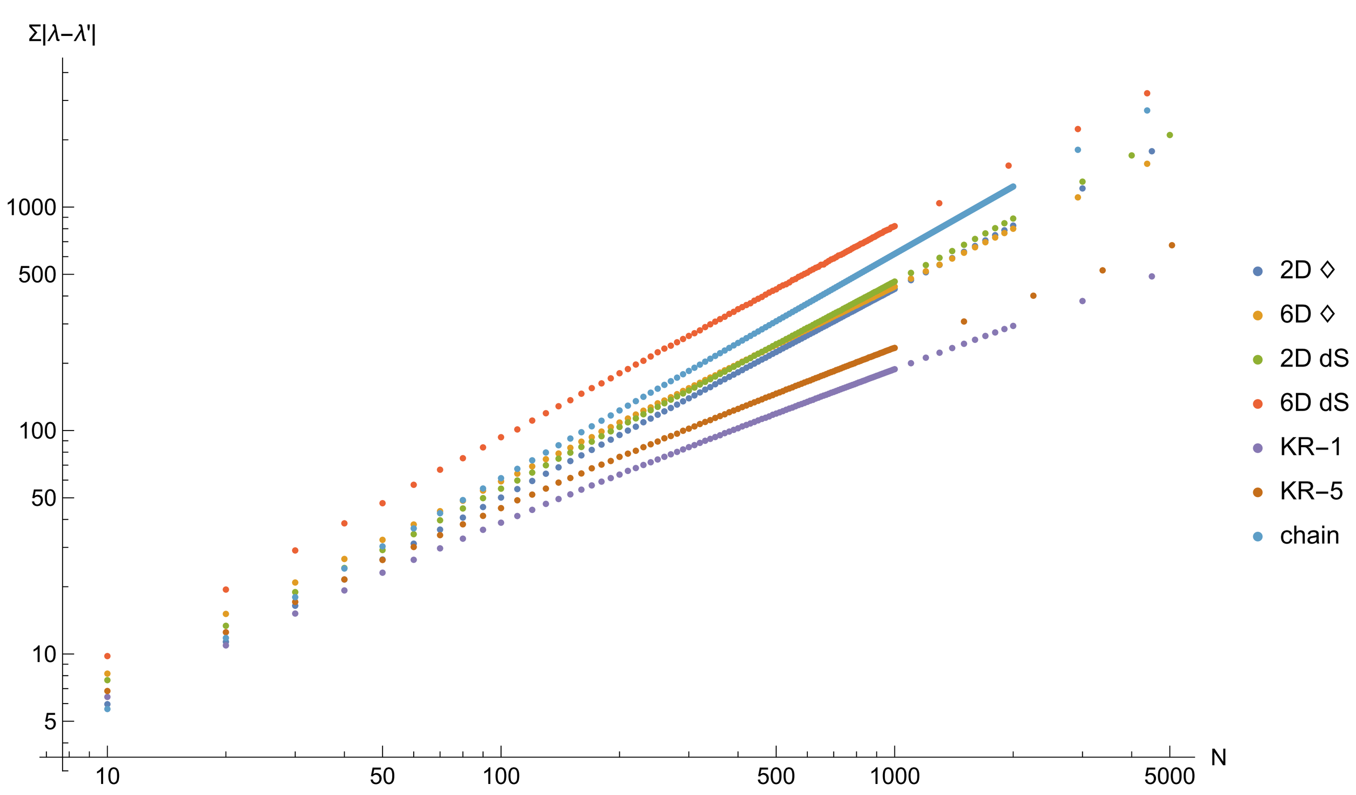

Figure 4. Log-log plot of spectral distance between original and perturbed causet against number of elements in the original causet. We are quill perturbing all of the elements of the causet. This process was done 20 times for each causet taking the average spectral distance. ◇ denotes a causal diamond, dS denotes a causet embedded in de Sitter space and KR-x denotes a KR order with x middle layers.

We perturbed every element of a causet and then calculated the spectral distance between the perturbed and original causets. In figure 4, we can see the dependence of this spectral distance on the size of the causet for different types. They seem to obey different power laws. In particular, non-manifold-like and manifold-like causets form two distinct groups.

Outlook

We have seen that power laws appear for discrete spacetimes, indicating that Weyl's law might apply more widely. We have also gathered significant evidence that manifold-like and non-manifold-like causets can be distinguished by their spectra and that dimension might be determined for causets above a minimum size.

References

[1] Luca Bombelli, Joohan Lee, David Meyer, and Rafael D. Sorkin. Space-time as a causal set. *Phys. Rev. Lett.*, 59:521–524, Aug 1987.

[2] Astrid Eichhorn and Sebastian Mizera. Spectral dimension in causal set quantum gravity. *Classical and Quantum Gravity*, 31(12):125007, May 2014.

[3] Mark Kac. Can one hear the shape of a drum? *The American Mathematical Monthly*, 73(4):1–23, 1966.

[4] D. J. Kleitman and B. L. Rothschild. Asymptotic enumeration of partial orders on a finite set. *Transactions of the American Mathematical Society*, 205:205–220, 1975.

[5] Yasaman K Yazdi, Marco Letizia, and Achim Kempf. Lorentzian spectral geometry with causal sets. *Classical and Quantum Gravity*, 38(1):015011, Dec 2020.