Causality of the Relativistic Scalar Field

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Theory and Background

The Scalar Field

Quantum Field Theory (QFT) describes a system in terms of a scalar field as a function of time and space. The simplest case involves a real scalar field S whose dynamics can be described in the form of a Lagrangian below.

$$\mathcal{L} = \frac{1}{2}\partial_\mu S \partial^\mu S - \frac{1}{2}m^2 S^2$$

However, S by itself holds little physical meaning. From $\mathcal{L}$, we can obtain the Klein-Gordon equation that describes the evolution of S below.

$$\left(\partial_\mu \partial^\mu + m^2\right)S = 0$$

Flaws of the Klein-Gordon

The fundamental flaw of the Klein-Gordon equation is that it is second order in both space and time. As a result, we get multiple obvious problems:

1. Negative energy solutions: Since our field is *real*, we cannot interpret these negative energies as antiparticles
2. Negative probability: The probability density ρ can be negative.
3. No conserved current: We cannot form a conserved probability four current J^μ which gives $\partial_\mu J^\mu = 0$.

A Revised Approach

To introduce a first order equation, we first transform S through the chain below as per [1].

1. Define the true field ϕ in terms of S as $\phi = \hat{H}^{1/2}S$.
2. Write down $\mathcal{L}$ in terms of ϕ

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \hat{H}^{-1}\partial^\mu \phi - \frac{1}{2}m^2 \phi \hat{H}^{-1}\phi$$

3. Define the wavefunction ψ as $\psi = \frac{\phi + i\hat{H}^{-1}\partial_t \phi}{\sqrt{2}}$

$$\hat{H}^n(r_i, r_i^{'}) = \int (m^2 + p^2)^{\frac{n}{2}} \frac{e^{ip_i(r_i - r_i^{'})}}{(2\pi)^3} d^3p$$

Where $\hat{H}^n(r_i, r_i')$ is a non-local operator which acts on the Fourier transform of a function in momentum space.

From the single particle wavefunction ψ linked with the true field ϕ, we can construct a conserved probability density and current as below.

$$\rho = \frac{1}{2}\left[\psi^*\psi + \left(\hat{H}^{-1}\hat{p}\psi\right)^*\left(\hat{H}^{-1}\hat{p}\psi\right) + m^2\left(\hat{H}^{-1}\psi\right)^*\left(\hat{H}^{-1}\psi\right)\right]$$

$$J_i = \frac{1}{2}\left[\left(\hat{H}^{-1}\hat{p}\psi\right)^*(\psi) + (\psi)^*\left(\hat{H}^{-1}\hat{p}\psi\right)\right]$$

Using ρ and J_i, we approach the test and proof of causality of ψ in the next section.

Study of Causality

Causality

For an approach to be truly relativistic, any system should be constrained by the speed of light. There are two ways to approach this question revolving around either the wavefunction or its evolution.

Wavefunction

Van Baal invokes the spread of probability over a light cone using S as a wavefunction [2]. This leads to the non-causal area shaded green in figure 1.

While this implies non-causality, there are two things to consider:

1. This argument interprets the initial condition on S as a Dirac delta wavefunction. ψ being the true wavefunction means that the initial condition is no longer trivial.
2. Hegerfeldt's theorem [3] shows that a positive-definite state cannot be confined as assumed in figure 1.

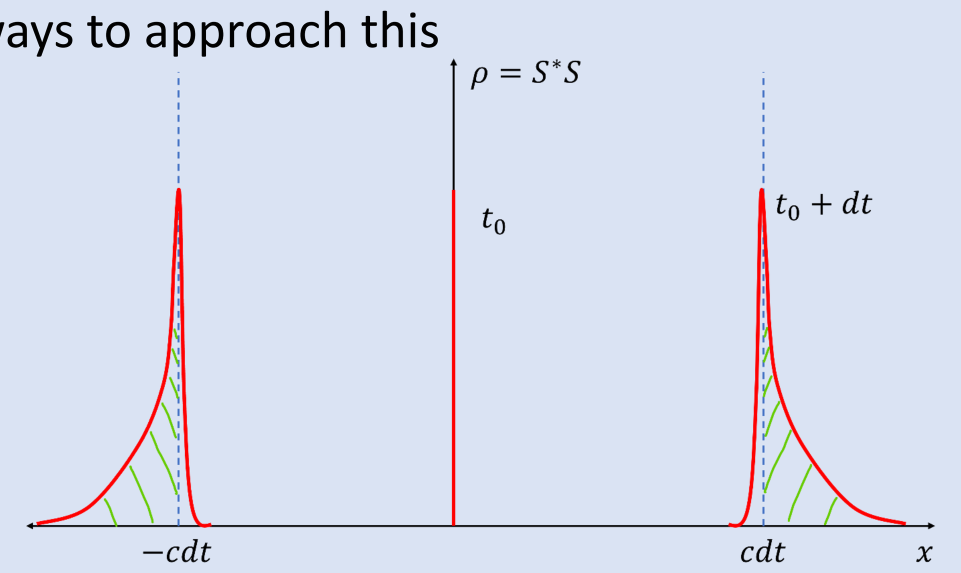


Figure 1. The wavefunction argument for causality violation for a delta function initially at 0 at time t_0 and spreading to a non-causal region shaded in green.

Evolution

Since information cannot travel faster than the speed of light c, we can instead develop a constraint using the evolution of the probability density given above of a general wavefunction ψ.

Asserting that the area under a general wavefunction in figure 2 is causal, we can conclude that the area shaded in green should be greater than the area in yellow. This gives us the inequality $\rho \geq |J_i|$ over all points and can be proven through the AM-GM inequality. Therefore, by this criterion, we can say that the wavefunction is causal.

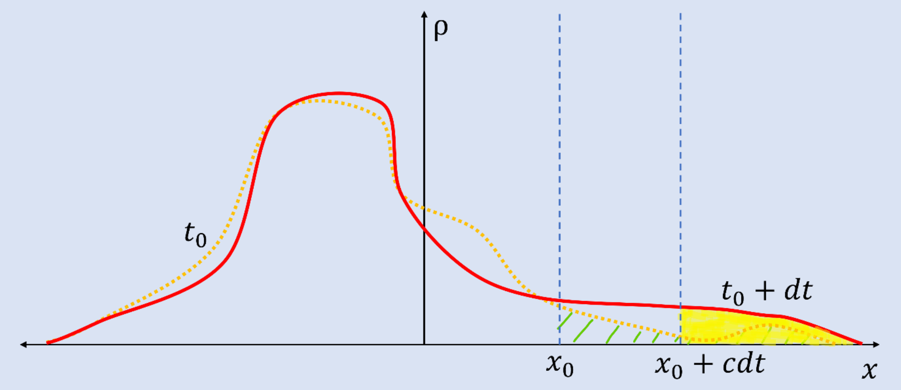
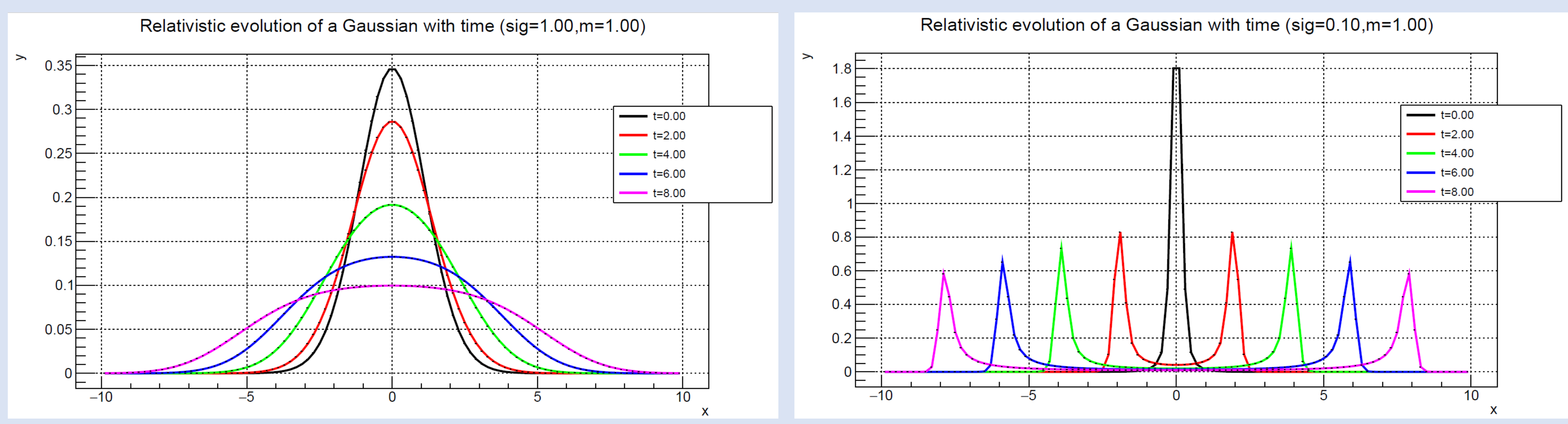


Figure 2. The evolution argument for causality violation for any ψ at time t_0 (orange) evolving through time dt (red).

Example: Gaussian

While we have demonstrated above that any wavefunction evolves causally, it is important to check whether it is physical. The best analytic option is to use a Gaussian to describe a particle of mass 1 GeV. The usage of natural units means every quantity is expressed in terms of GeV units. We choose to compare widths of 1 and 0.1 GeV^{-1} for this study.

Figure 3 shows this comparison over an 8 GeV^{-1} time interval. The behaviour of the 1 GeV^{-1} case appears to be more classical with the spread while the 0.1 GeV^{-1} case demonstrates two tails as consistent with relativity.



(a) $\sigma = 1$ (b) $\sigma = 0.1$

Figure 3. A comparison of the successive evolution of ρ over space for a Gaussian width of 1 and 0.1 GeV^{-1}. The time steps in GeV^{-1} are 0, 2, 4, 6 and 8 respectively.

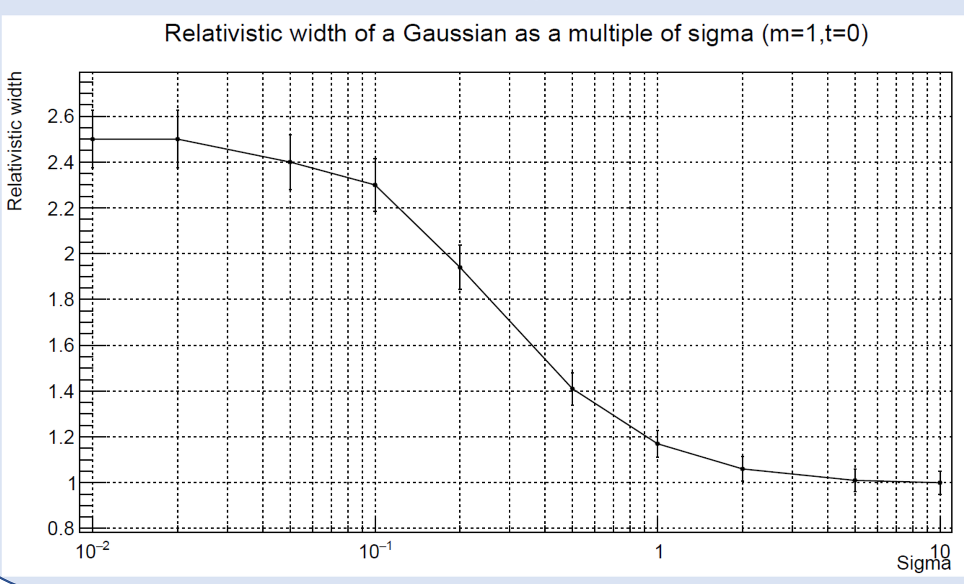


Figure 4. An expression of the relativistic width of ρ with respect to the width of $\psi^*\psi$ as a scale factor. The error bars account for the imprecision in fitting a Gaussian to a non-Gaussian ρ and the x-axis is set to be logarithmic.

The extent of deviation from the classical Gaussian was also measured. As displayed in figure 4 to the left, as ψ becomes narrower and more relativistic, the RQM probability density widens.

Conclusion

This project has effectively shown that a Relativistic Scalar Field interpreted as ϕ yields a causal wavefunction ψ which appears to be physical for a Gaussian.

However, there are still open questions to be addressed when it comes to the physical origin of the probability density:

1. Can we derive the ρ and J_i analytically by requiring local gauge invariance of $\mathcal{L}$?
2. Are ρ and J_i together a four-vector J^μ which transforms under the Lorentz group?
3. Does $\mathcal{L}$ require the addition of total derivative terms to obtain a J^μ?

Once these challenges are met, we can extend this study to even Complex Scalar Fields.

References

[1] P. Dauncey, *Relativistic Quantum Mechanics - Lecture 10*. Imperial College London, Oct 2020.

[2] P. Van Baal, *A Course in Field Theory*. Taylor & Francis, 2013.

[3] G. C. Hegerfeldt, "Causality, particle localization and positivity of the energy," *Lecture Notes in Physics*, p. 238–245.