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Rigidity of Faithful Embeddings in Causal Set Theory

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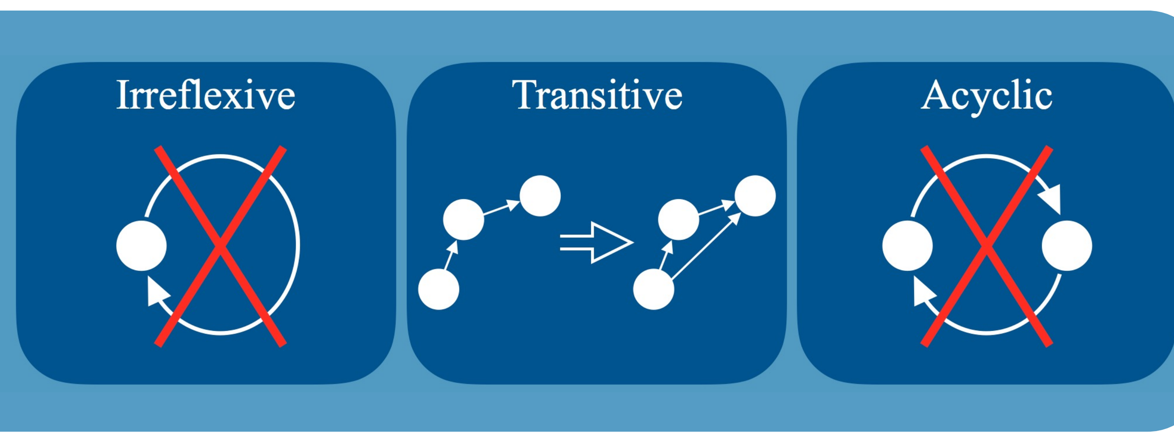
Introduction to causal set theory

Causal set theory is one of the leading theories aiming to explain the features of quantum gravity. It postulates that space-time is described by causal sets at its finest scale. A ***causal set*** is a locally finite set with partial ordering on its elements. The elements correspond to events in the continuum, while the ordering encodes the causal structure [1].



Glossary

Poisson process: a random process that positions points so that the density of points in any finite region follows a Poisson distribution.

Faithfulness: a causal set is a *faithful* representation of a manifold if the manifold could have risen from it as a continuum approximation.

Sprinkling: a technique to obtain causal sets that are faithful representations of a given manifold. Points in the manifold are generated via a Poisson process and used as elements of the causal set. Then every element is endowed with the order it inherits from the causal order of the manifold.

Embedding: given a manifold and a causal set that faithfully represents it, we can assign a coordinate to every element in the causal set [1].

Our approach

We aim to thoroughly investigate the properties of faithful embeddings. We are analysing the limitations of recovering the continuum properties of a space-time from its causal set. Concurrently, we are investigating the rigidity of an embedding and the global effect of coordinate fluctuations to its causal structure.

Embedding into a 1+1D Minkowski diamond

Once we know that a causal set is a faithful representation of a 1+1D Minkowski diamond, we can attempt to construct an algorithm to embed it.

Assuming a uniform density distribution over the diamond, we expect the number of points between any two elements to be proportional to the volume of intersection of their future and past light cones (equal to the square of the proper time between the two elements). We can then develop a sequential technique based on this observation.

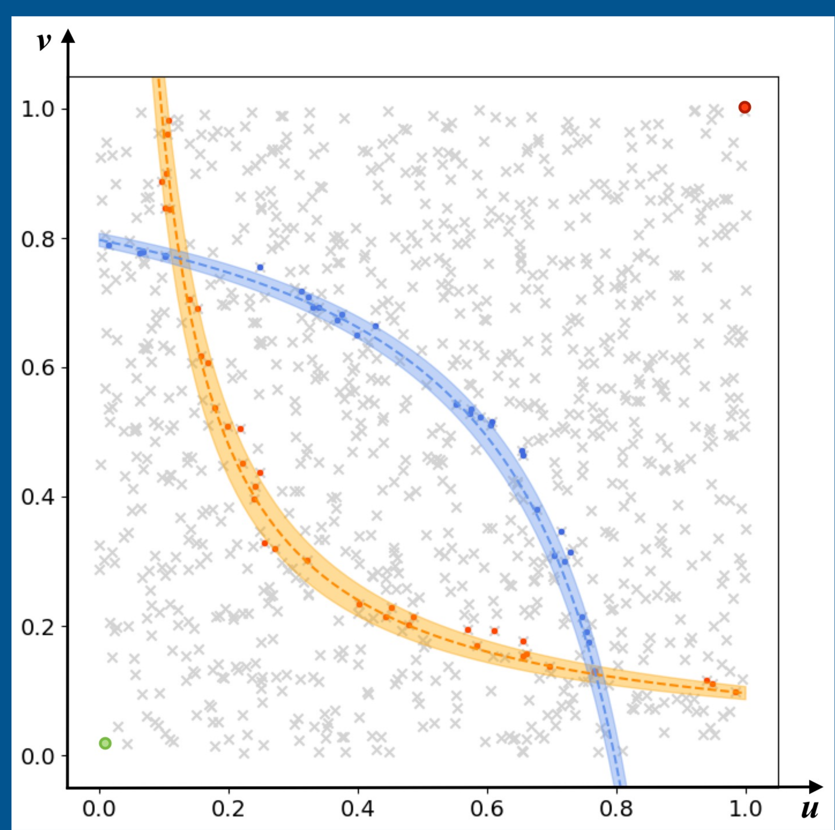


Fig 1 – Visualization of estimating the coordinates. The orange/blue region has a set past/future size (± ε). Their intersection gives our coordinate estimate.

First, we find the maximal and minimal elements (those with every other element to their past/future), then we estimate the position of every element of the causal set, knowing the proper time from the minimal point to the element as well as the proper time from the element to the maximal point (see Fig 1). This embedding technique has previously been described [2, 3], however our analysis revealed some flaws with this method when estimating the position of elements close to the centre.

The *Scaffolding* approach

Our solution for this phenomenon was to reconstruct the continuum by starting at the edges and working inwards. Once all the elements close to the edges of the diamond are calculated, we can use these to find the position of the elements in the middle. As seen in Fig 2, the inclusion of this technique significantly improves the quality of the embedding.

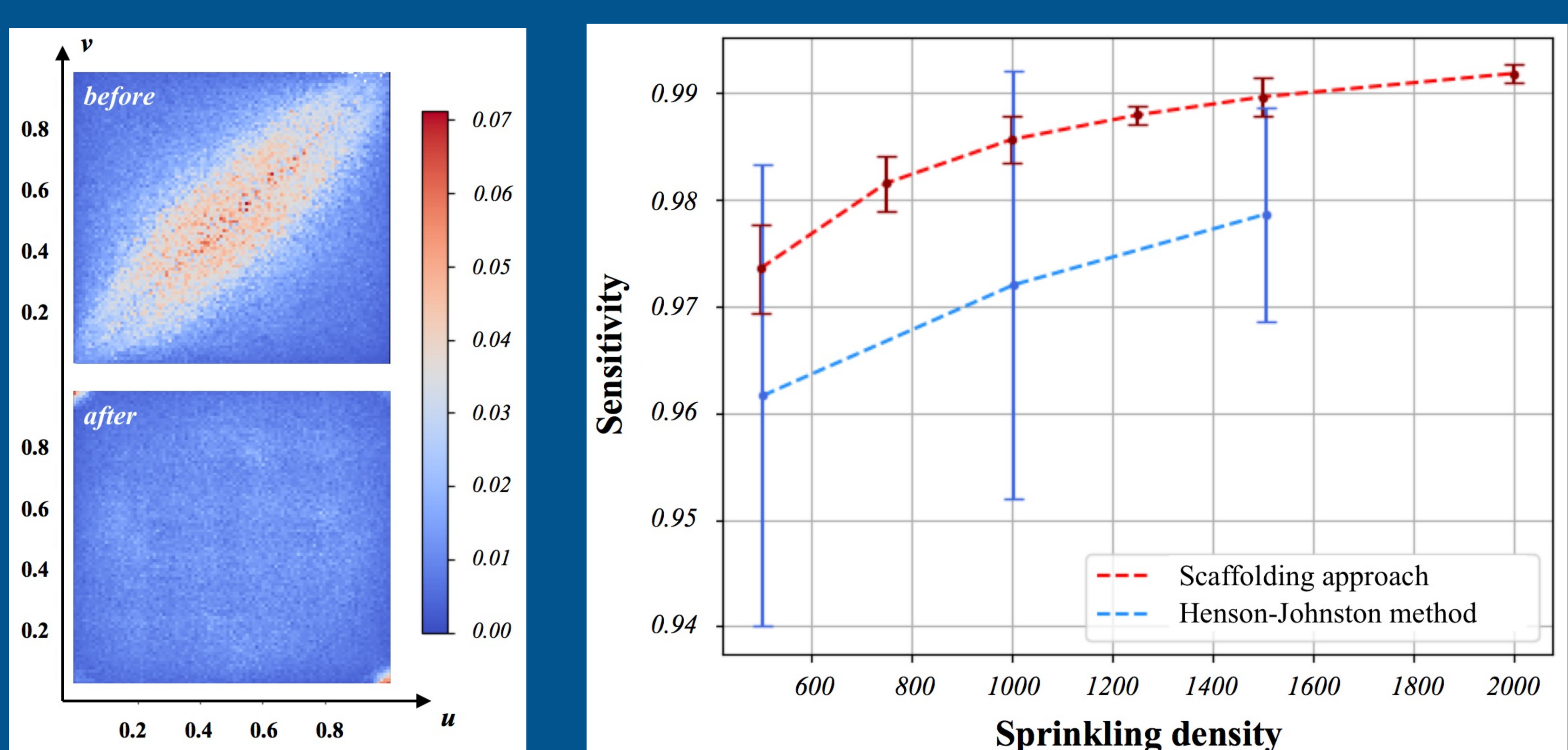


Fig 2 – (left) visualization of the error in the coordinate estimation before and after the Scaffolding approach; (right) a comparison plot showing the fraction of accurately preserved relations (sensitivity) by each embedding method as the sprinkling density increases. The error bars represent the standard deviation over 10 embeddings.

Rigidity scale of an embedding

For a causal set faithfully embedded into a Minkowski diamond, we investigate how much the element coordinates can be shifted without changing the causal order between them. This allows us to probe the rigidity of the embedding and access how a local fluctuation can have a global effect in the causal ordering.

An iterative approach

To approach this, we constructed an algorithm where we surround each element in the embedding with a sphere of radius δ, vary its coordinates in a random angular direction such that it lies on the surface of the sphere (see Fig. 3), and recompute the causal relations.

We then iteratively decrease the size of the sphere until the causal ordering is the same as in the initial embedding. The critical value of δ at which this happens defines the rigidity scale of the embedding.

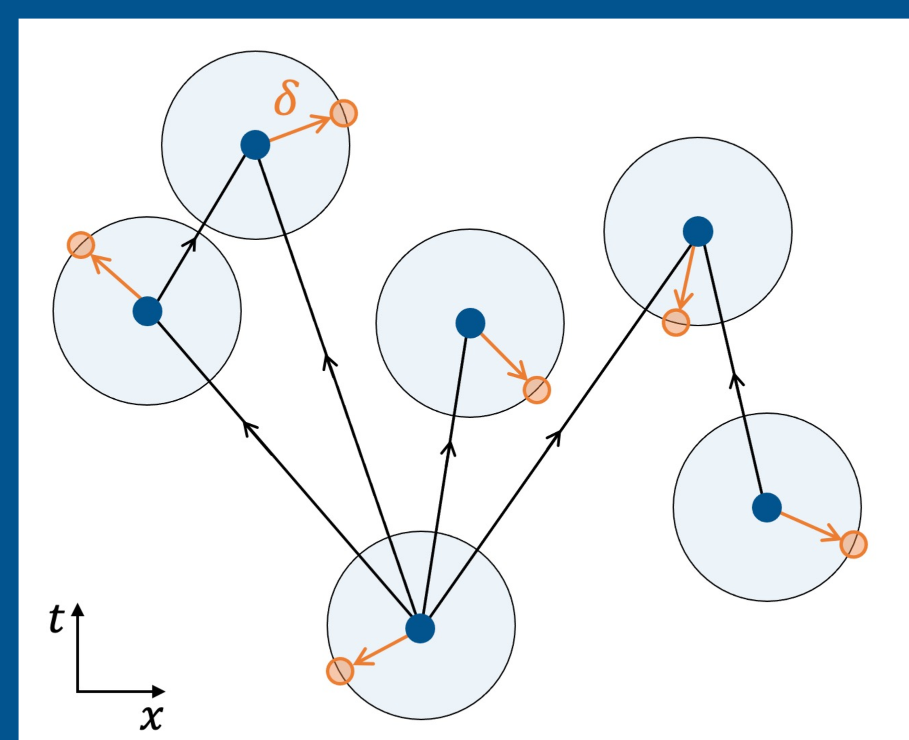


Fig 3 – Diagram illustrating our method to vary the coordinates of the elements in 1+1D. The black directed lines connecting the events (blue dots) represent the causal ordering. Orange dots represent their coordinates after the variation.

Analytically, we have proven that the spheres cannot overlap, meaning that δ is constrained by half the shortest distance between two elements in the embedding. In 1+1D, we have also calculated δ analytically. Both of these calculations were used to verify our iterative method.

Conclusion and further work

Our investigations demonstrate that causal sets are able to encode the continuum geometry of manifolds in their order structure. This is inline with the theory and supports the causal set theory approach to quantum gravity.

Now we are working on extending our work to higher dimensions as well as invoking recent developments of artificial intelligence to further improve the quality of our embeddings. Concurrently, we have started investigating how δ varies with the density of elements in the embedding.

References

[1] Rafael D. Sorkin, *Causal Sets: Discrete Gravity,* pages 305-327. Springer US, 2005
[2] Joe Henson, *Constructing an interval of Minkowski space from a causal set,* 2018
[3] Steven Johnston, *Embedding Causal Sets into Minkowski Spacetime.* 2021