A Generalized Method For Finding Hidden Symmetries In Dynamical Systems

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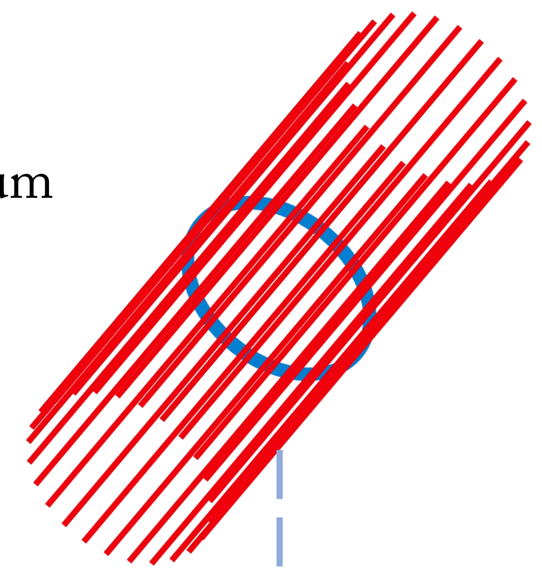
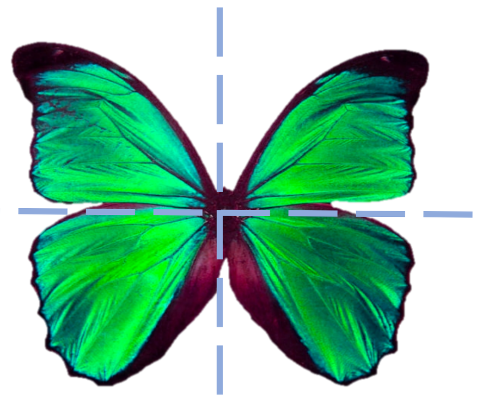
Motivations + Objectives

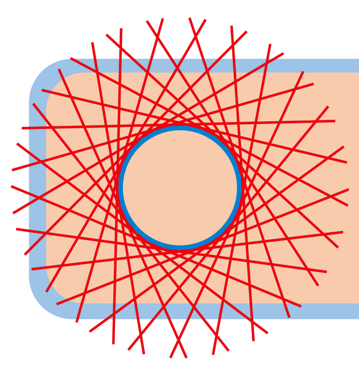


- Initially motivated by attempts to understand *"accidental"* degeneracy in quantum systems
- A well known example is SO(4) hidden symmetry which explains the angular momentum quantum number *"accidental"* degeneracy in the hydrogen atom
- This lead to the objective of finding a generalised method for obtaining symmetries in any dynamical system
- Attempt to show that a differential-geometric approach simplifies the search for hidden symmetry



What is Symmetry?

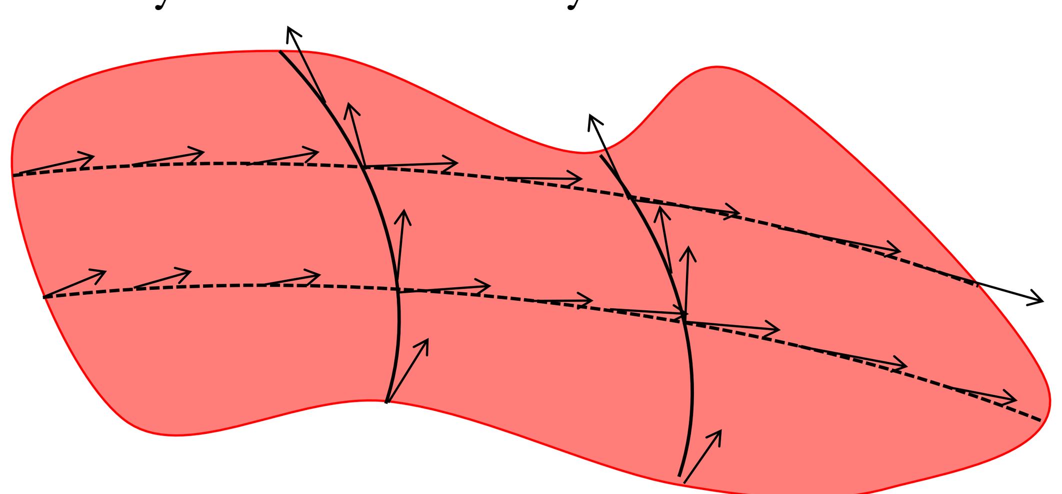
- Symmetry is the invariance of objects and/or equations under some transformation
- Geometric symmetries correspond to invariances of the configuration of the system and are usually intuitive
- However, there exist other symmetries corresponding to conservation of abstract quantities through system dynamics – called dynamical symmetries (often *"Hidden"*)



Continuous symmetries of Hamiltonian systems can be studied by considering the natural symplectic geometry of the problem.

Why Symplectic geometry?

- Hamiltonian systems naturally live on a symplectic manifold where each point is a phase space $\mathbf{x} = (\boldsymbol{q}, \boldsymbol{p})$ with a symplectic structure ω that endows it with a Poisson structure
- Transformations of the system then correspond to flows $\gamma(t, \boldsymbol{x_0})$ generated by vector fields $\hat{X}_G$
- These flows are lines on the phase space that obey an ODE with initial condition $\boldsymbol{x_0}$ at $t = 0$
- Flows that commute with the Hamiltonian flow are symmetries of the system



Machinery

- Geometrical symmetries correspond to **Killing vectors** such that the vector field $\widehat{\boldsymbol{X}}_{\boldsymbol{G}}$ along the configuration space only depends on position $\boldsymbol{q}$ .These are quantities that obey specific properties depending on the geometry
- Dynamical symmetries correspond to higher-order **Killing tensors** such that $\widehat{\boldsymbol{X}}_{\boldsymbol{G}}$ along the configuration space now depends on both positions $\boldsymbol{q}$ and momenta $\boldsymbol{p}$
- For more complicated problems the analysis can be simplified by considering a higher dimensional manifold whose geodesics are the Hamiltonian paths (*Eisenhart lift procedure)*
- For example, the hidden symmetry of the Hydrogen atom corresponds to a rank-2 **Killing tensor:**

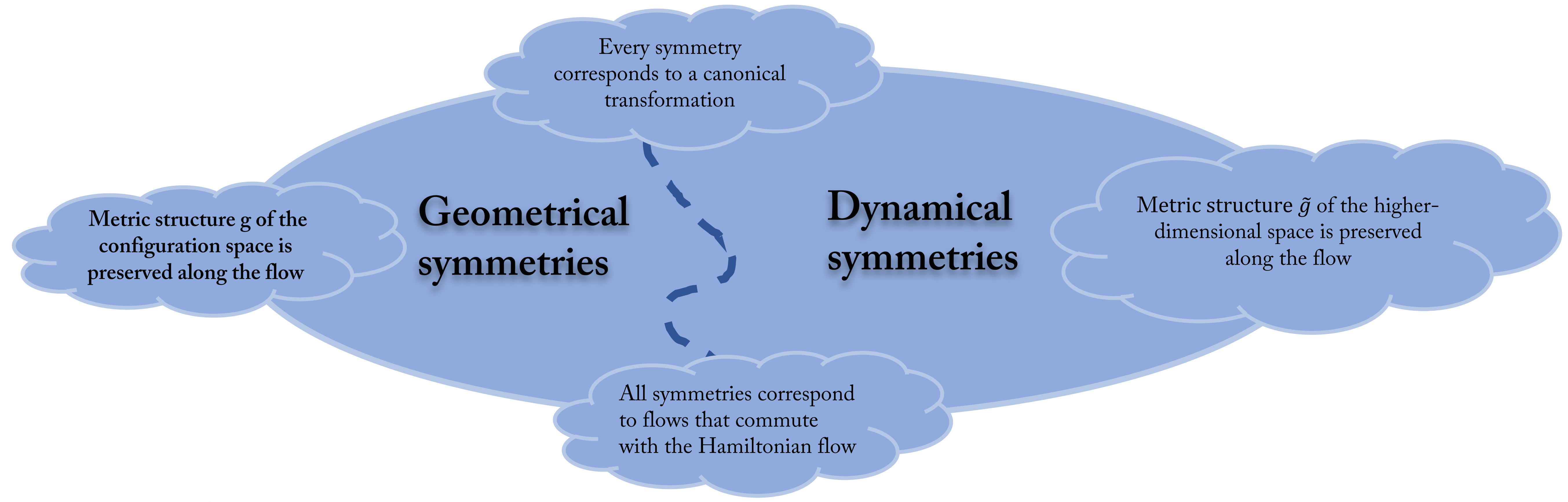
Method found in literature: complicated and speculative

$$\delta q^i = \dot{q}^i q^k a_k - \frac{1}{2} q^i \dot{q}^k a_k - \frac{1}{2} q^j \dot{q}_j \delta^{ik} a_k$$

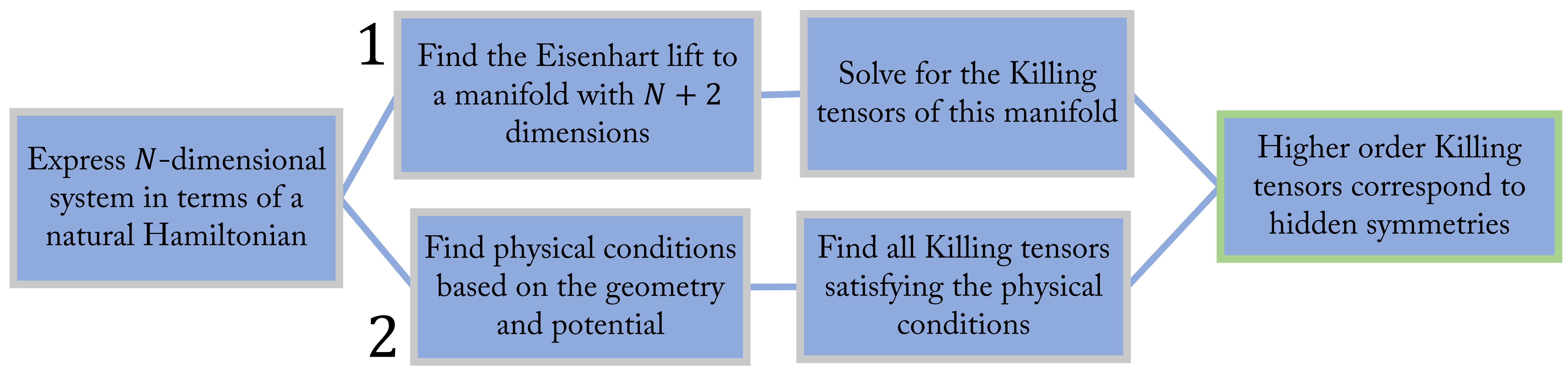
Killing methodology: simple and systematic

$$\kappa^{ij} = a^i q^j + a^j q^i - 2\delta^{ij} a_k q^k$$

In Summary…



Proposed Methodology



Extensions: Kerr Metric, Supersymmetry, Kaluza-Klein Theories, Inflation Models, RIP accidental