Analysis of resonant frequencies of the double-pendulum

Michal Horanský

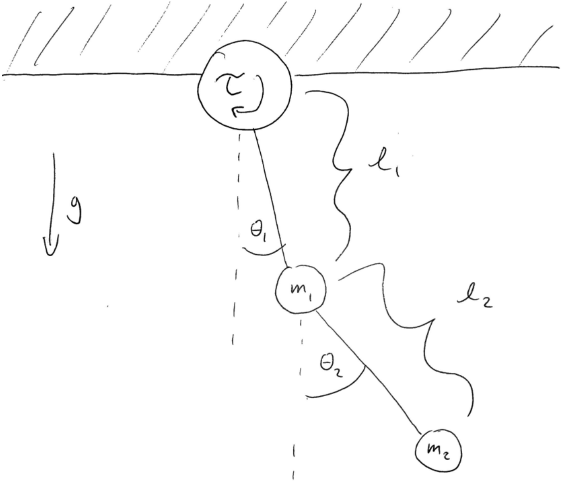
Imperial College London

February 9, 2022

Introduction and motivation

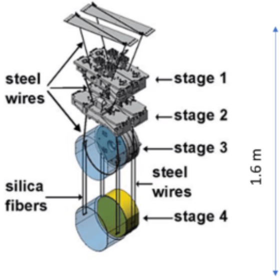
We've got the system to [study] the system.

Aphrodite's Child, 1972 (paraphrased)

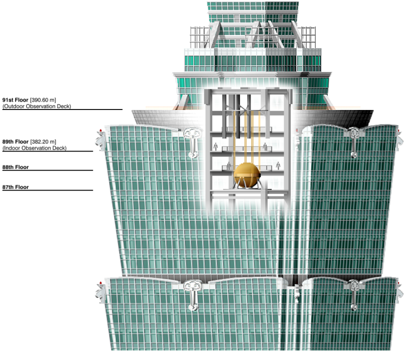


Introduction and motivation

LIGO's Quadruple Pendulum



The top two masses of the LIGO pendulum are 20 kg and the lower two masses 40 kg.



Images credit: LIGO/Caltech; Wikimedia/Someformofhuman

The motion of the driven double-pendulum

1. Torque acting on the upper segment: $\tau = l_1 F \sin \omega_F t$
2. Equations of motion are as follows:

$$
\begin{aligned}
\dot{\theta}_1 &= \omega_1 \\
\dot{\theta}_2 &= \omega_2 \\
\dot{\omega}_1 &= \frac{F \sin \omega_F t - m_2 \sin(\theta_1 - \theta_2)(l_2 \omega_2^2 + l_1 \omega_1^2 \cos(\theta_1 - \theta_2))}{l_1((m_1 + m_2) - m_2 \cos^2(\theta_1 - \theta_2))} \\
&\quad - \frac{g(m_1 \sin \theta_1 + m_2 \cos \theta_2 \sin(\theta_1 - \theta_2))}{l_1((m_1 + m_2) - m_2 \cos^2(\theta_1 - \theta_2))} \\
\dot{\omega}_2 &= \frac{F \sin \omega_F t \cos(\theta_1 - \theta_2) - (m_1 + m_2) l_1 \omega_1^2 \sin(\theta_1 - \theta_2)}{l_2 (m_2 \cos^2(\theta_1 - \theta_2) - (m_1 + m_2))} \\
&\quad + \frac{-m_2 l_2 \omega_2^2 \cos(\theta_1 - \theta_2) \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \cos \theta_1 \sin(\theta_1 - \theta_2)}{l_2 (m_2 \cos^2(\theta_1 - \theta_2) - (m_1 + m_2))}
\end{aligned}
$$

What's interesting?

> *There's definitely (...) no logic to [double-pendulum] behaviour.*
>
> Björk, 1993 (paraphrased)

1. General motion is chaotic
2. Interesting restriction: small driving torque ($F \ll g(m_1 + m_2)$) when starting from a stationary state
3. Why is this interesting? Remember real-life models

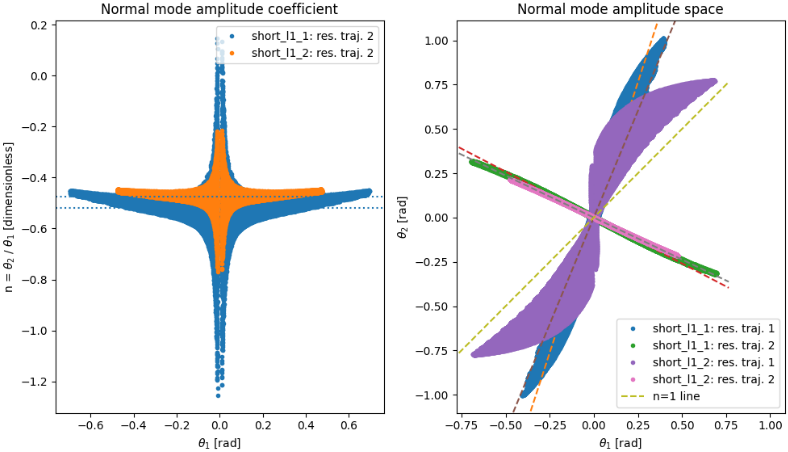
Resonance

1. Simple to define for a simple pendulum
2. What is of interest to us for the double-pendulum?
   2.1 Maximal amplitude of the upper segment over a long time period?
   2.2 Time limit of the total mechanical energy accumulated? Note the complex energy dissipation
3. Our goal: find all ω_R for the double-pendulum for which resonance emerges
4. Two proposed definitions virtually indiscernible; I chose maximal θ_1
5. Possible connection to normal modes?

Normal modes of the double-pendulum

1. Simply put: solutions where $\tilde{\theta}_2(t) = \tilde{n} \cdot \tilde{\theta}_1(t)$
2. Usually correlates with resonant frequencies (empirical)
3. We need to show validity of restricted solution of the equations of motion
4. Trivial solution: $n = 1$ ("phase solution")
5. For phase solution: $\omega_R = \sqrt{\frac{g}{l_1+l_2}}$ - notice invariability to many properties
6. Observed "antiphase solution" with nontrivial $|n|$
7. Empirically: ω_R for phase solution approaches triviality for $l_2 = 0$, ω_R for antiphase solution approaches triviality for $l_1 = 0$

Normal modes of the double-pendulum



1. Phase solution much better behaved for slightly lower $\omega_F \rightarrow$ imperfect correlation

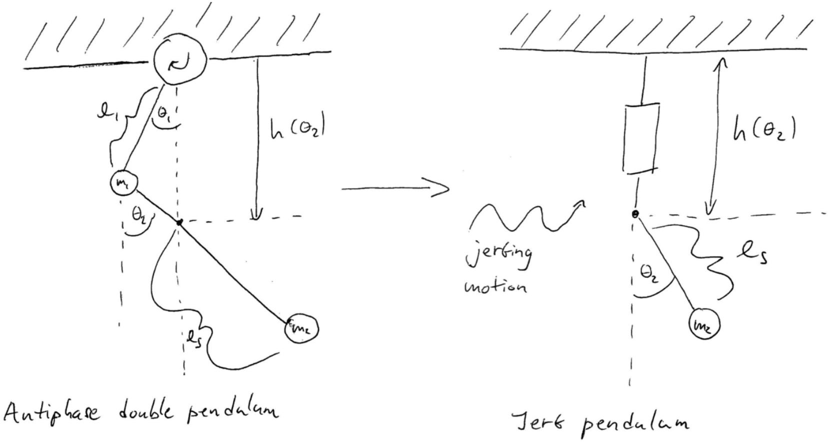
The fulcrum trick

You [double-pendulums] have such cute [resonant frequencies].

Toshiro Mifune, 1961 (paraphrased)

1. Consider the phase solution - what other system does this represent? What connection is there to its natural frequency?
2. Can we find a similar representation transformation for the antiphase solution?

The fulcrum trick - antiphase solution



$$l_s = l_2 - l_1 \frac{\sin\theta_1}{\sin n\theta_1} \approx l_2 - \frac{1}{n} l_1;\ h(\theta_2) \approx l_1 (\cos\frac{\theta_2}{n} + \frac{1}{n}\cos\theta_2)$$

The fulcrum trick - antiphase solution

1. In the non-inertial frame: fulcrum as the reference point

$$L = T - U = \frac{1}{2} m_2 l_s^2 \dot{\theta}_2^2 + l_s m_2 \cos\theta_2 g', g' = g - \frac{\mathrm{d}^2 h}{\mathrm{d}t^2}$$

2. Using the E-L eqn. and with Taylor expansion omitting $O(\theta_2^3)$:

$$\frac{g}{l_s}\theta_2 + \ddot{\theta}_2 + \epsilon\left[\ddot{\theta}_2\left((m+1) - \frac{m}{2}(1-m^2)\theta_2^2\right) - \dot{\theta}_2^2\theta_2(m^3+m+2)\right] = 0$$

where $\epsilon = m\frac{l_1}{l_s}$ (minusculity justified by edge case)

3. By linear perturbation analysis, omitting $O(\epsilon^2)$:

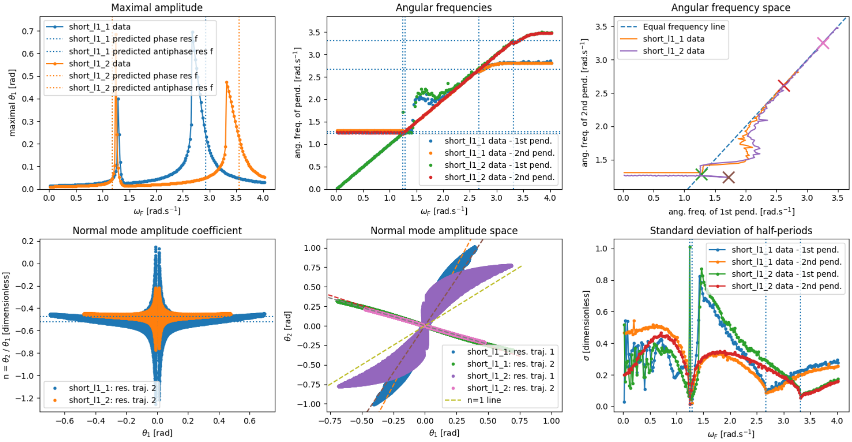
$$\begin{aligned}\theta_2 \quad &= \quad \tilde{A}\left[e^{i\omega_0 t} + m\frac{l_1}{l_s}\left[e^{i\omega_0 t} + \frac{\tilde{A}}{16}(m+1)(m^2-m+4)\left(e^{3i\omega_0 t} - e^{i\omega_0 t}\right)\right.\right. \\ &\qquad \left.\left. - i\omega_0 \frac{m+1}{2} t e^{i\omega_0 t}\right]\right]\end{aligned}$$

where $\omega_0 = \sqrt{\frac{g}{l_s}} = \sqrt{\frac{g}{l_2 - l_1/n}}$

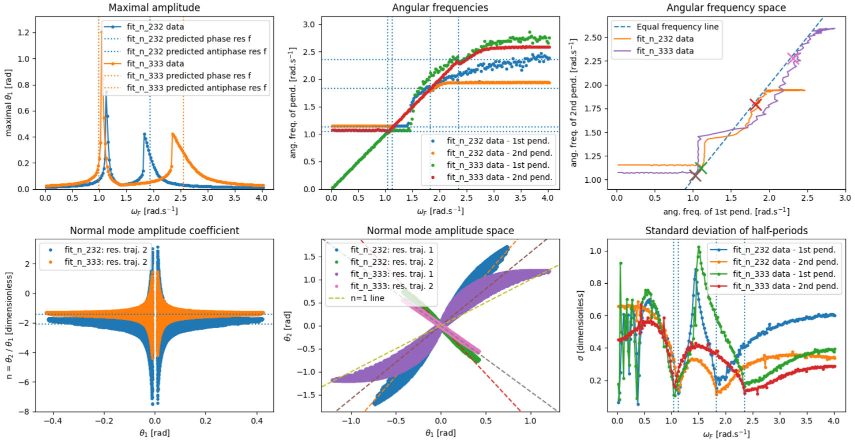
The fulcrum trick - antiphase solution

1. Taylor expansion turned pendulum into SHO$\rightarrow$ we expect dynamic rescale in actual solution
2. Dominant term in the long term: the one proportional to t - this can be shown to be true for the whole solution by examining the cross-terms (we can't obtain another repeated root of the characteristic equation)
3. Corresponding natural frequency: $\omega_R = \omega_0 = \sqrt{\frac{g}{l_2 - l_1/n}}$
4. We'll find this a good approximation for the antiphase resonant frequency
5. Note the small l_1 assumption
6. Note that n is not an intrinsic property - needs to be observed

Actual behaviour - short l_1, changing m_2



Actual behaviour - $l_1 = l_2$, changing m_2



Conclusions and further research

1. The two predicted resonant frequencies - $\sqrt{\frac{g}{l_1+l_2}}$ and $\sqrt{\frac{g}{l_2-l_1/n}}$ - are good estimates of actual resonant frequencies (difference always under 15%)
2. We've empirically shown that antiphase resonance is modal and phase resonance is almost modal (notice even half-periods)
3. Empirical groundwork for correcting these estimates
4. Further research questions/topics:
   4.1 Find the antiphase solution of the equations of motion using found restrictions to express ω_R only using intrinsic properties
   4.2 Study the angular frequency space outlier phenomenon
   4.3 Try extending these results to find average angular frequencies of both segments for arbitrary ω_F

Acknowledgement

1. The K3 group: Marlene Koerner, Miguel M. Nalda, Rian Patel, Alvin Peng, Binbing Sun
2. The incredible Jarek Ciba
3. My family and my personal tutor for their support

Thank you for your attention