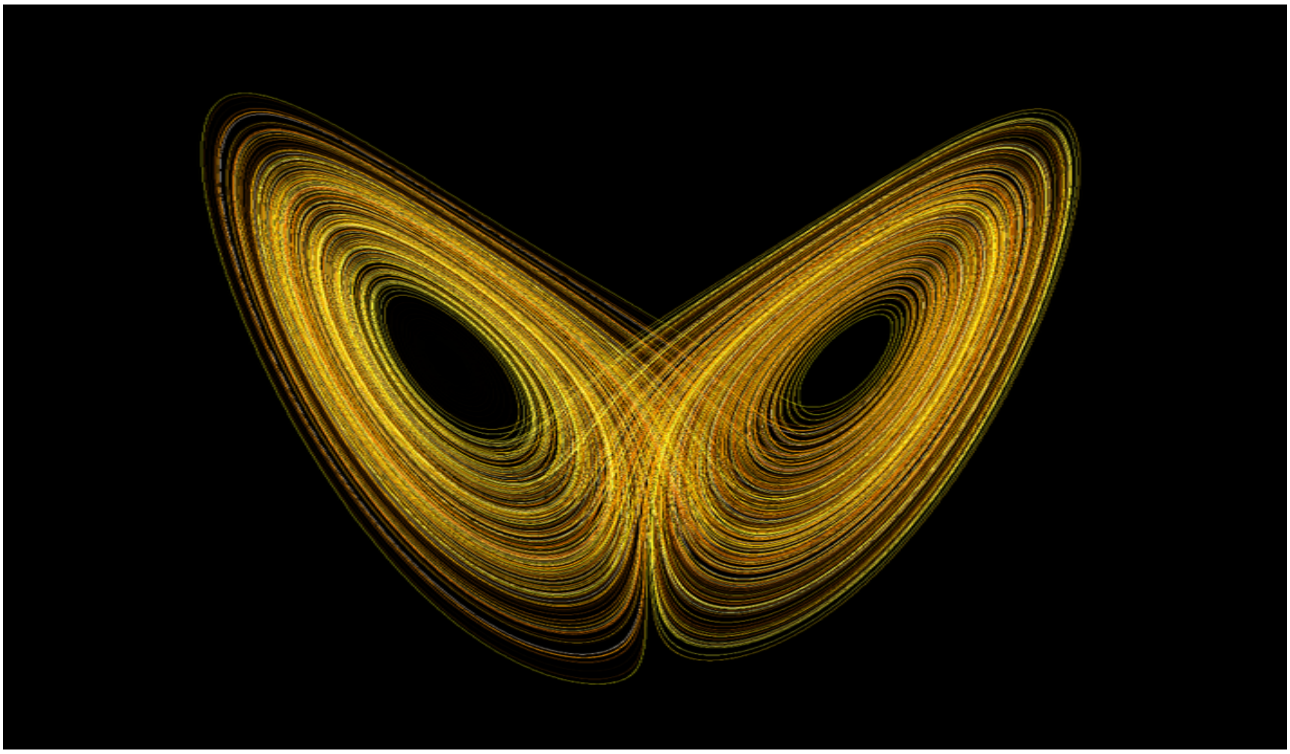

Quantum Chaos: quest for a superimposing order rooted in uncertainty

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Imperial College London Second Year Quantum Article
Word Count: 2635
Cover Photo depicting Lorenz attractor (Halpern, 2018)

Quantum chaos: quest for a superimposing order rooted in uncertainty

Challenges to determinism and the birth of a new era in physics

For centuries, the world of physics was reined by the determinism of the laws of Newton and his contemporaries. In this seemingly perfect universe where systems follow deterministic patterns, if you know the complete initial conditions of a system, you can predict everything about how it will behave in the future. The brink of the twentieth century brought with it two very significant directions in physics that deemed Newtonian determinism an essential yet incomplete description of the universe - chaos theory and quantum mechanics. This article will try to give the reader an appreciation for what is in essence a merger of these two fields, which have transformed our understanding of our universe. The development of this has been a culmination of the works of many great scientists, from Kepler to Einstein, and brings together some of the most important concepts in modern physics (Hart-Davis, 2012).

The story of chaos dates back to the times of Kepler and Newton and to the fundamentals of all physics – orbiting planets. The motion of two orbiting bodies is well defined and has been formulated eloquently by Newton's law of gravitation, yet when a third body is added to the system, it gains complexity very quickly and solutions that satisfy the law can be exponentially unstable (Ullmo & Tomsovic, 2014). In the late 19th century, French physicist Henri Poincaré's seminal work revealed that "most dynamic systems show no discernible regularity or repetitive pattern" (Gutzwiller, 1992). Orbits of members of the solar system, such as asteroids, comets and dust particles, belong in the category of such systems, displaying chaos and instability as they move in the gravitational fields of larger bodies like planets which cause perturbations in their trajectory on million-year time scales (Malhotra, Holman & Ito, 2001).

The unpredictability within order – when billiards becomes more than just a game

We refer to chaos in our daily life to describe situations that lack order or organization. In physics and mathematics, we define chaos as "the extreme sensitivity in a system to changes in initial conditions" (Rudnick, 2008). A simple yet excellent way to depict chaos, which we will later extend to other domains of physics, is to examine the motion of billiard balls on tables of different shapes. Think of a flat billiard table without any pockets in shapes seen in Figure 1. We assume that there is no friction between the ball and the surface of the table. Say, the ball is given an initial impulse and starts moving in an arbitrary direction. It will bounce from the sides of the table, reflecting off with an angle equal to the angle of incidence. The ball will continue in moving in this fashion, as it eventually spans out the entire billiard table. If we then repeat the same procedure with the ball with an ever so slightly different starting point, we will see that the shape of the trajectories will be different for tables shaped differently. While the difference between the two trajectories represented by blue and black lines in Figure 1 will remain very small for the circular table (Figure 1(a)), in the stadium shaped table (Figure 1(b)) and the cardioidal table (Figure 1 (c)) the trajectories diverge significantly (Ullmo & Tomsovic, 2014). Stadium and cardioid billiards thus display sensitivity to

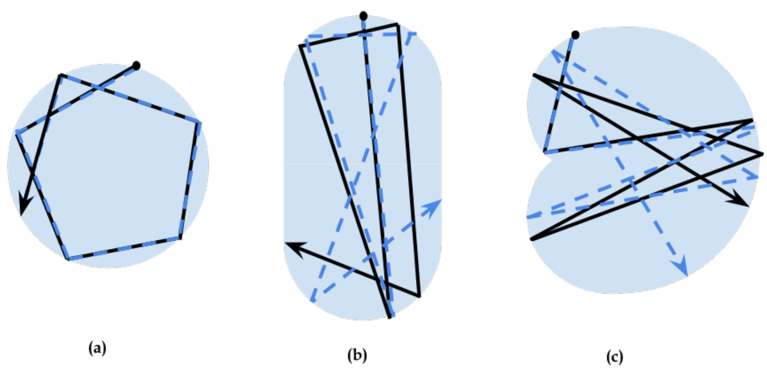

Figure 1: Chaos arising in billiards. The trajectories for two billiard balls with slightly different initial positions in tables shaped as a (a) circle, (b) stadium and (c) cardioid.

initial conditions and are chaotic systems where infinitesimal differences are amplified as the system evolves in time.

We often observe chaos in dynamical systems with many degrees of freedom, although even simpler systems such as the stadium and cardioid billiards described can exhibit chaos. One of the major motivations to study chaos came from studying the atmosphere and the differential equations that describe it when physicist Edward Norton Lorenz discovered how great of a divergence it could cause to run his atmospheric model with initial conditions that are only three decimal places different (Hart-Davis, 2012). Figure 2 depicts a map which was drawn by Tímea Haszpra using chaos theory to predict paths of particles emitted into the atmosphere with wind data (Carne, 2019). Chaos is omnipresent in nonlinear dynamical systems of our universe, ranging from the structure of galaxies to population growth of species (Borwein & Rose, 2012). Chaotic systems are interesting in that, despite their prohibition of sensitive prediction, they follow a perfect deterministic set of equations ticking like clockwork (Borwein & Rose, 2012). It is interesting to witness chaos in systems of very different inner workings.

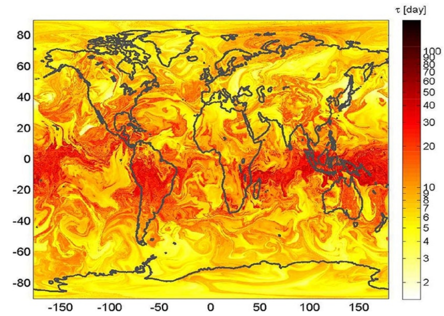

Figure 2: Map showing predicted path of emitted particles in the atmosphere drawing using chaos theory from the work of Haszpra (Carne, 2019)

So how can we identify a classically chaotic system? The main tool we can use to identify the degree to which the chaos is restricted is the Kolmogorov-Arnold-Moser (KAM) theorem. This allows us to calculate how much of the structure of a regular system survives in the presence of a small perturbation and can thus allow us to identify perturbations that result in chaotic behavior (Gutzwiller, 1992). Another powerful tool is the Lyapunov exponent, a measure of the rate of the exponential separation with time of initially close trajectories (Gan, 1996). If a system is chaotic, it will yield a strictly positive Lyapunov exponent. There is still much research being conducted on chaos, in both purely mathematical forms and applications, and it is recognized as one of the most important phenomena in physics.

New directions in chaos

The idea of a completely deterministic universe was no longer valid after the developments in quantum mechanics, which was built on the works of many great physicists such as Planck, Einstein, Bohr, Dirac and more in the 20^{th} century. It was soon noticed that light isn't just a wave, the atom doesn't follow the model of a moon orbiting a planet and things on a subatomic scale can behave rather strangely. Niels Bohr formulated the Correspondence Principle in 1913 with his atomic model, which was later found to be inaccurate (Dhar, 2016). The Correspondence Principle, however, still survives and states that in the limit of large quantum numbers, quantum mechanics reproduce classical mechanics. This motivates the search of phenomenon we observe in classical mechanics on a quantum mechanical scale. The question that arises naturally is whether it is possible to find chaos in quantum mechanical systems. Since quantum mechanics underlies classical mechanics, shouldn't there be some kind of analog to chaos on a quantum scale? How can the extremely irregular character of classical chaos be retrieved from the smooth and wavelike nature of quantum systems (Gutzwiller, 1992)?

The Bottomline – Quantum suppresses chaos

To answer such questions, we must look into fundamental quantum mechanical concepts. In quantum mechanics, a system is described by a 'wave function' – complete mathematical description of the system's state. The wave function is governed by the Schrödinger equation (Eq.1), a partial differential equation that can be said to be the quantum counterpart of Newton's second law of

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{\mathcal{H}}|\Psi(t)\rangle$$

Equation 1: Schrödinger's Equation. Gives the time evolution of a wave function in quantum mechanics.

motion. It gives the evolution of a wave function over time. In the quest for quantum chaos, one approach to take is to ask whether two slightly different wave functions diverge exponentially from one another with time. The short answer to this is no – the linearity of Schödinger's equation, the fact that a linear combination of two solutions to it will also be a solution – makes it impossible that chaos emerges in such way in quantum systems (Ullmo & Tomsovic, 2014).

Another essential of quantum mechanics is Heisenberg's Uncertainty Principle, as expressed in Eq.2. According to this, there is a physical limit to the degree of certainty to which you can know the position and momentum of a quantum mechanical system. These two quantities are inversely proportional - knowing position of a particle with higher certainty entails less accuracy in the momentum. This renders the concept of a trajectory in quantum mechanics inaccurate, as it's impossible to know the positions the particle traces with certainty. If a trajectory isn't traceable in the first place and it's impossible to make infinitesimally small changes and observe the sensitivity of the trajectory to the initial conditions, one can't speak of chaos in the classical sense.

$$\Delta x \Delta p \geq \hbar/2$$

Equation 2: Heisenberg's Uncertainty Principle. Gives the fundamental limit of certainty we can have about the position of a particle x and its momentum p.

So how would a quantum system with a classically chaotic counterpart act? This was tested by experimentalists in the 70's and 80's. In 1979, Casati and his collaborators experimented with a kicked quantum pendulum, a system that would act chaotically in the classical counterpart (Blümel, 1994). If chaos works the same way on a quantum scale, it would be expected that complicated quantum behavior, such as exponential sensitivity to changes in the wave function, would arise from this system. Such behavior was not found, as the system showed no sensitivity to the wave functions. The conclusion was that quantum interference effects suppress chaos in wave functions of quantum systems (Blümel, 1994).

So, what's all the research about?

This may seem like quantum chaos is at a dead end. If the Schrödinger equation and Heisenberg Uncertainty Principle entail quantum systems don't display sensitivity to initial wave functions, then why study quantum chaos? Turns out the problem of quantum chaos can be approached in a different way to reveal valuable information about quantum systems. Chaos makes itself apparent in quantum systems in other ways (Rudnick, 2008). Let's phrase the problem as trying to understand whether the properties of quantum analogs of classically regular systems differ from those of classically chaotic systems (Bunakov, 2016). This is the direction quantum chaos research has taken and was given the name "quantum chaology" by the pioneering physicist Michael Berry. He defines quantum chaology as the "study of semiclassical, but nonclassical, phenomena characteristic of systems whose classical counterparts exhibit chaos" (Berry, 1989). When quantum and wave systems in the semiclassical limit are studied, it is possible to see classical chaos make itself apparent in the distribution of energy levels of different systems, which we will now explore (Bunakov, 2016).

Ideas that Arise From Chaos - Revisiting Billiards in quantum scale

To explore further some of the quantum signatures of chaos, let's imagine a quantum scale analog of the billiards system, which we saw earlier, was chaotic classically. There are many ways in which we can do this – a particular one is to use a microwave cavity. This is a closed metal structure that confines microwave electric fields. The microwaves will bounce back and forth between the walls of the cavity and will form standing waves at resonant frequencies (Stöckmann, 1999). These standing waves are the analog of a ball bouncing off the edges of a billiard table. These are called the eigenmodes of the system, and can be obtained from the minima of a plot of frequency against reflected power. At first glance, it is difficult to see what useful information this spectrum can give, however a surprising result arises when the probability of finding a specific spacing between adjacent energies corresponding to the eigenmodes is plotted, as seen by Stöckmann's histogram in Fig. 3 (1999). The expected energy spacing of the classical counterpart of this system follows a Poisson distribution, of the form $P(s) = \exp(-s)$, as seen by the dotted line in Fig. 3. However, in the quantum system it is possible to see significant divergence from this as the probability of finding very close energies is very low. The distribution that best fits the pattern is Wigner's distribution, of the form $P(S) = \frac{\pi}{2} S \exp(-\frac{\pi}{4} S^2)$, as seen by the smooth line in Fig. 3 (Stöckmann, 1999).

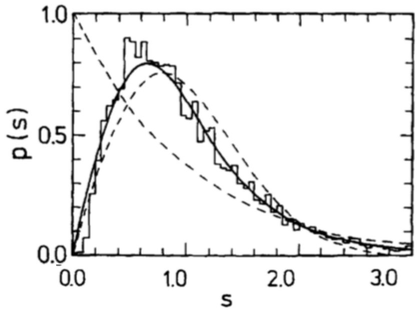

Figure 3: Nearest neighbor distance histograms for a microwave cavity for frequencies in range 15 to 18 GHz obtained by Stöckman (1999).

The Hydrogen Atom in a Strong Magnetic Field

Now let's consider a different system. One way to explore the territory in between classical and quantum mechanics is to study the behavior of a hydrogen atom in a strong magnetic field. In order to take the hydrogen atom in a strong magnetic field to the semiclassical limit, it is excited to very high energies, where it exhibits classical properties. This is called the Rydberg atom. The trajectory of the excited electron is highly scattered, indicating chaotic behavior (Gutzwiller, 1992). If we study the probability distribution of the energy levels, we can see that it is very different from the random distribution that a classically nonchaotic equivalent, such as a molecular hydrogen ion displays. Plotted in Figure 4, we can see that it is very unlikely to find two energy level that are very close together, showing that there is some kind of repulsion in between the energy levels in the chaotic Rydberg atom (Gutzwiller, 1992). Once again, the data does not follow a Poisson distribution which would be expected in the nonchaotic equivalent. Furthermore, the chaotic spectrum fits Wigner's distribution, the same pattern the microwave billiards discussed earlier. This resemblance reveals one of the most intriguing features of quantum chaos - its identical traces in completely different systems. It turns out that this statistical distribution is present in the spectra of many chaotic systems (Ullmo & Tomsovic, 2014).

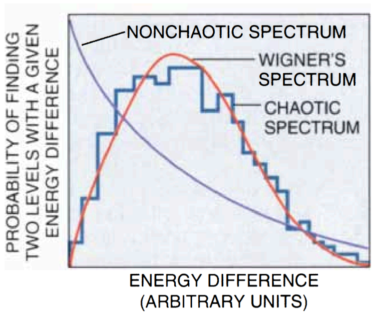

Figure 4: Gutzwiller's (1992) plot of the probability distribution of differences in energy levels in the Rydberg atom.

Quantum chaos has taken a path to reveal patterns about such statistical spectra, which show parallels in many systems in nature. However, this is perhaps ambiguous from the title "quantum chaos", as we have discussed that quantum chaos isn't simply chaos arising in quantum systems in the classical sense. Furthermore, we have discussed ways to distinguish classically chaotic systems, but it is clear that doing this in quantum systems is not straightforward. In order to end the ambiguity, in his 2016 paper, Russian physicist Bunakov suggested another metric for the definition of chaos based on symmetries rather than trajectories and derived a quantitative measure of chaos, which is calculated based on perturbations breaking symmetry in chaotic systems (Bunakov, 2016).

So why should we care?

In physics we treasure the idea of patterns that can be extended to any system, no matter the size. It is exhilarating to see different systems behave identically, leading to an element of universality that quantum chaos poses. Such systems with parallels include acoustic wave intensities found in scattering problems known as the Rayleigh distribution, Ericson fluctuations in neutron scattering and conductance fluctuations in quantum dots (Ullmo & Tomsovic, 2014). The omnipresence of similarities in statistical spectra hint that there is little sensitivity to the specifics of these systems and a more general framework that rules their behavior exists. If satisfaction in understanding such dynamics itself isn't enough to convince the reader that chaos and quantum signatures of chaos are worth studying and researching, we can propose the information they help reveal in other many domains, such as nuclear physics, acoustics and certain materials, as an acceptable reason (Stöckmann, 1999).

A few words to conclude…

Chaos is fascinating. It presents us the idea that something that works entirely based deterministic equations can grow into something unpredictable. The search for such dynamics in quantum systems, whose working underlie the whole of classical physics was only natural following advances in quantum chaos. Quantum chaos isn't only a concept – it's a field of study that has had many evolving theoretical and experimental directions over the years, corroborating each other and giving direction to the future of the field. In studying the quantum counterpart of classically chaotic systems and comparing them to non-chaotic quantum systems, patterns indicating chaos, such as statistical spectra following the same distribution was found in systems of very different forms, two of which have been discussed. The quest for a superimposing order rooted in uncertainty continues to adapt to new findings and reveals valuable information about the workings of quantum and wave systems.

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Article Plan:
Quantum Chaos – how quantum mechanics superimposes its own order in the chaos

The aim of this article will be to introduce the idea of quantum chaos. Considering the breadth of the field, the article will not try to provide a comprehensive review of the questions posed by, but rather it will try to outline how chaos theory and quantum mechanics are intertwined, with emphasis on how this is obtained from physical phenomenon. In essence this is the merge between two of the most fascinating ideas in physics- chaos theory and quantum mechanics- and its development has been a culmination of the works of many great scientists, from Kepler to Einstein. This article will try to give the reader an appreciation for this.

Introduction: (~500 words)

Introducing chaos: To begin with, what is chaos and how do we observe it classically? Exemplifying chaos via orbits. What does it mean in the macroscopic world and is it relevant to quantum mechanics?

What about quantum chaos?: What are the motives to study quantum chaos-why do we care? Give context on the historical turn of events that led to the study of quantum chaos. (Gutzwiller, 1992)

Chaos arising in quantum phenomena: (~700 words)

Billiards: Use the example of billiards to explain the evolution of a system into chaos. Show the extension to quantum mechanics. Talk about quantum billiards experiments in various scales (microwaves inside a cavity and electrons), including in optical systems. Comment on the implications of these experiments. (Gubin & Santos, 2012)

The hydrogen atom in a strong magnetic field: Explanation of how chaos is apparent in the energy levels of the hydrogen atom. (Ullmo & Tomsovic, 2014)

The Bottomline – Quantum suppresses chaos: (~700 words) Under what conditions is chaos suppressed by quantum mechanics? (Rudnick, 2008) What is the quantum break time? Quantum break time in experimental contexts – double kicked atoms

Ideas that arise from the study of quantum chaos: (~700 words) How is chaos reflected in quantum systems in other ways?

- Chaos in random matrix theory and the statistics of the energy spectrum.
- Talk about how these ideas can be applied in acoustics.
- Applications in chaotic scattering

Conclusion – the "So What?": (~400 words)

Summarize main points in the useful information that arises from the study of a classically chaotic system with quantum mechanics

General outlook: Briefly mention the current directions of research relating to quantum chaos.

Recap the charm of quantum chaos as the overlap of fascinating physics

Out of the scope of this article:

- details into the mathematics of chaos theory
- the full historical development of the field (the turn of events leading to the discussion of quantum chaos amongst scientists will be briefly mentioned)
- details of random matrix theory
- details of Gutzwiller trace formula

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Feedback from Academic Tutor:

I liked that you emphasize the historical significance/context in the outline, that with resonate with pop sci readers. I think quantum chaos will be an interesting topic. The challenge will be that quantum and chaos are both challenging to explain to a non-expert reader. You will need to be careful you make sure to give the reader enough jargon-free explanation to follow your story. The billiards/quantum billiards example looks very engaging. I like the sound of the quantum suppresses chaos and ideas arising from… sections. I do worry that the plan looks very ambitious and you are planning to take on a lot of different things. I emphasize that it is better to do a few things well than a lot of things badly! If when it comes you writing you find yu need to cut some things I would start with 'The hydrogen atom in a strong magnetic field' I would keep the following technical aspects to a minimum

details of random matrix theory

details of Gutzwiller trace formula

You have found some good looking references. In the final article make sure you use proper journal citations and not just weblinks.

Response to feedback:

Many thanks for the feedback. I have decided to remove some sections that are too technical and focus on explaining the fundamental concepts well. As I started writing the article I realized it takes a lot more words to explain certain things, therefore I simplified my initial plan and took out certain things that don't necessarily fit in the storyline. I decided to go with a longer introduction to chaos, as I believe there is value in explaining it even in classical sense and remove some of the "ideas that arise from chaos" from my article. By removing these sections I hope to add more emphasis to things I do write about and have more space to explain them properly. Overall, I wanted to keep the focus of the article on the concept of chaos and quantum signatures in chaos.