The Fundamental Reflections of the Universe

Word Count: 2974

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Symmetries are more fundamental to our understanding of physics now than ever before. They have helped physicists conjecture theories such as Super Symmetry and test some of the most fundamental laws in nature.

Figure 1: Symmetry can be seen all around us. M.C. Escher was fascinated by reflections in spheres, physicists today are interested in the reflections and rotations in space and the consequences of these transformations [1].

Symmetries are all around us, from Escher's famous spheres in figure 1 to the crystalline structure of diamond. They define a regularity in nature and a predictability in our everyday lives. Yet these seemingly innocuous properties of systems have a much deeper meaning for physicists and the fundamental laws of physics. These symmetries in the laws of nature lead to the conserved quantities that physicists treasure and use almost every time they do a calculation. Yet the importance of these properties had been ignored for centuries.

The symmetry laws underpinning classical mechanics had not fully been investigated until Einstein's Special Theory of Relativity in 1905, in which he considered symmetries as a fundamental fact of nature, and that these symmetries constrain the physical laws that can arise [2]. This shift towards using symmetries to restrict the laws of nature that could exist was intrinsic to Einstein's development of the General Theory was still considered a quirk of relativity and group theory. However, Wigner and other physicists pioneered the use of the method in quantum mechanics to answer problems, such as the theoretical atomic spectra of more complicated atoms [3], which would be almost impossible to solve using standard techniques.

The most basic symmetry law, that almost all physicists know, is the *time reversibility* of the Classical laws. That is, if we know what is happening now we can predict what will happen in the future by evolving the system according to our laws, but we can also learn about the past be reversing this evolution. This *time reversibility* leads directly to the conservation of energy, one of the most fundamental laws of physics. More investigation into these symmetries reveals the origin of other conserved quantities such as linear and angular momentum. While the symmetry laws in classical physics are incredibly useful, their use in quantum mechanics leads to some very strange and insightful results, which will be explored later on. Nevertheless, we first must look at the mathematics behind this strange area of physics and so we must take a brief look at the abstract mathematics that is Group Theory.

A Crash Course in Groups

Groups are of fundamental importance in analysing symmetry laws in physics, they are the natural representation of these symmetries because of the way quantum mechanics can be formulated. This section will set the theory of this area out in a fairly formal manner, I've tried to give an example so that the concept becomes clear. The group is built up from a set $G = \{a, b, c...\}$ which has some additional structure. The $\bullet$ is just an operator of any kind. It could represent matrix multiplication, addition, subtraction or something more complicated. The operation depends entirely on the set G and what kind of group is to be formed.

A group G is defined by the four following properties [4]:

1. It is closed, that is if $a, b \in G$ then $a \bullet b \in G$. Which basically means that two objects acted together produces an object which is still in the set.

2. It is associative, that is if $a, b, c \in G$ then $a \bullet (b \bullet c) = (a \bullet b) \bullet c$

3. It has an identity e such that if $a \in G, a \bullet e = a$

4. Each element has an inverse, that is for $a \in G$ there exists $a^{-1} \in G$ such that $a \bullet a^{-1} = e$

$$G = \left\{ e_n | e_n = \begin{pmatrix} \cos(n) & -\sin(n) \\ \sin(n) & \cos(n) \end{pmatrix}; n = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \right\}$$

	e_0	$e_{\frac{\pi}{2}}$	e_π	$e_{\frac{3\pi}{2}}$
e_0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$e_{\frac{\pi}{2}}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
e_π	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$e_{\frac{3\pi}{2}}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Figure 2: An example of a contracted matrix group, where the operator $\bullet$ is matrix multiplication. The objects in the set G are the right-angle rotations about the xy-plane, we can see that the identity is the identity matrix ($\mathbb{1}$), each element can be multiplied by another element to produce $\mathbb{1}$ and so there are inverses. The most notable thing is, however, there is clear symmetry in the group which is the foundation of this area of physics.

This all seems very abstract at first glance, and that is actually the reason this area of mathematics is so useful for analysing symmetries. The symmetries we want to derive are elusive and typically can be very abstract and based in the heart of quantum mechanics.

The most likely set of objects used in this area of physics are matrices or exponentials phasors ($e^{i\alpha x}$) , and so the $\bullet$ (pronounced "blob") is usually the multiplication associated with those objects. There are groups that are very specific to symmetry in physics, and they are typically the Unitary and Special Unitary groups [4] which are basis for a lot of the global and local symmetries. Most of these groups are continuous and so they obey another part of group theory called Lie (pronounced 'Lee') algebra, the mathematics of these kinds of groups are not very important but there are a host of books on this area [5]. An example of a rotation group can be seen in figure 2.

The Constants in Our Lives - Global Symmetries

Global Symmetries are symmetries which affect the entire system and transforms the entire system homogeneously. This type of symmetry is the reason we have conservation of energy and momentum, and the entire idea behind this type of symmetry is summed up in Noether's theorem [6]:

> "Covariance of the equations of motion with respect to a continuous transformation with n parameters implies the existence of n conserved quantities ('conserved charges' or 'integrals of motion'), i.e. it implies conservation laws."

The three most basic transformations that we can think of are translation in time, space and rotation in space. These three transforms leave the equations of motion unchanged and so we can conclude from Noether's theorem that there are conserved quantities associated with these three translations [6]. Translation in time leads to the conservation of energy, the translation in space leads to the conservation of linear momentum and rotation in space leads to the conservation of angular momentum [1]. This is a fairly astounding result, because some of the most fundamental laws in physics are derived directly from the symmetries and covariance of systems under certain translations and rotations. What is even more amazing is that these transformations can be organised into mathematical groups. That is, the rotations in space form a Lie group and so do the other previously mentioned transformations. We can actually show if a set of transformations form a group, then a covariance in the equations occurs and thus we can find a conserved quantity associated with that transformation [7].

In quantum mechanics, however, we can multiply a system, ψ, by a phasor, $e^{i\phi}$, and this will still be a solution to the Schroedinger wave equation. Obviously ϕ can take on any value, provided $\phi \in \Re$, and so the group will be subject to Lie algebra, as suggested in the last section. However, just by inspection we can see it satisfies the four group properties. Of course $e^{i0} = 1$ is the identity, and so the inverse of an element is dictated by the rule that $\phi + \phi^{-1} = 0$. The other two properties can be easily proved too. As a result, we can conclude that this set of transforms, $\psi \mapsto e^{i\phi}\psi$, forms a group[2]. The fact it forms a group, we know, means there is an associated conserved quantity, which is a pretty amazing result given that all we considered was a basic transformation and from this we can deduced a conserved quantity.

[1]All of these results are fairly easy to derive but require looking at more advanced mechanics.
[2]This group is the U(1) group and plenty of information on this group can be found in the references of this article.

The conserved quantity associated with this group is charge, in the situation above, where $\phi \in \Re$ and $\phi \neq \phi(x)$, the charge conservation is a global conservation. If $\phi = \phi(x)$ then the phase would differ from point to point in the system and so would be said to be locally symmetric.

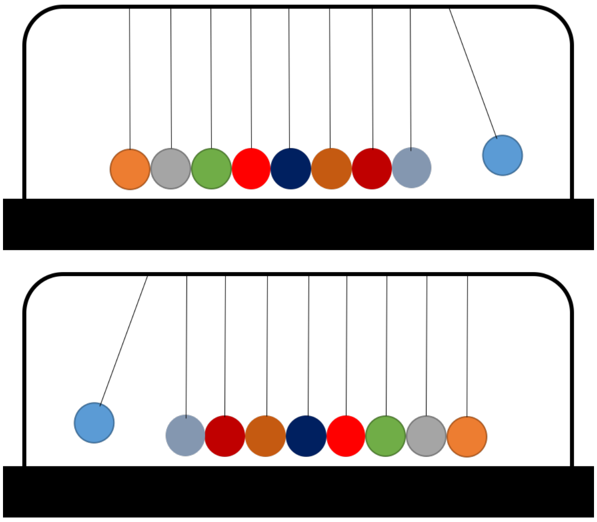

Figure 3: An example of a global symmetry. Here a Newton's cradle has been reflected (akin to the parity transformation) but we expect the period of oscillation to be the same in both systems, this is a rather contrived example but it serves well to demonstrate what a global transformation is and how it leads to conservation.

This is the first step we're going to make in finding something called the Charge-Parity-Time (CPT) symmetry which underpins the entirety of the standard model. The next thing to look at is the P, or parity, conservation of a quantum system.

Parity inversion is just replacing x with $-x$, see figure 3, in the wave equation or matrix of states, this either leaves the wave function unchanged (even) or reflects the system in the y axis (odd). The easiest way to prove that this operator, Π, forms a group is to use the matrix representation of the parity inversion operator, which has the form [8]:

$$\Pi_{inversion} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{1}$$

when you compose this with the $\mathbb{1}$, Π forms a group. The symmetry associated with this group is the conservation of parity, that is, if a state is odd it will remain odd for all time and if it is even it will remain even for all time. So now we know that parity is conserved, the final symmetry to find is time symmetry and then we'll have CPT symmetry, the underpinning for the standard model of particle physics and the Yang-Mill's theory. The proof of time symmetry is very involved, and is far beyond the scope of this article, however Wigner's original paper [9] and online notes [10] show the proof.

We now have the symmetry of charge, parity and time individually. However, when applied to the nuclear weak interaction these individual symmetries are violated due to the presence of massive bosons [11]. This was a huge blow for the symmetry community as it seemed to throw all the work Wigner and his colleagues had done aside. Fortunately, Chen Ning Yang and Robert Mills were working on an extended version of much of the work that had been previously completed, and eventually Yang-Mill's theory was born [12]. This theory lead QED (quantum electrodynamics), electro-weak unification, and later to unification with the strong force (QCD or Quantum Chromo-dynamics). The most important result, which we have to an extent walked through, is that taken together charge, parity and time are not violated by the weak interaction. CPT symmetry is the very reason we can draw Feynman diagrams and consider an electron evolving forward in time as a positron travelling backwards in time, see figure 4.

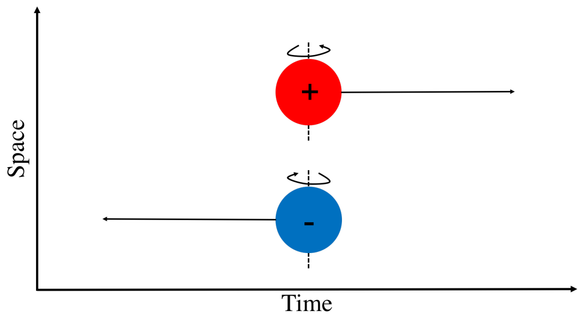

Figure 4: The top particle is a positively charged, forward evolving, "positive" parity particle. The bottom particle is exactly the opposite of this, however in Yang-Mill's theory these two particles can be considered the same, preserving CPT symmetry in QED and QCD.

Whilst CPT is a global symmetry, gauging (or using local symmetries) provides the full basis for QED and QCD, these theories rely on defining a gauge and then showing invariances in these gauges. It is the general consensus that global symmetries are all either broken or approximate symmetries to a more local symmetry which is almost homogeneous [2], this is because Wigner's symmetries seem to, "smell of action at a distance" [2]. So now we take a look at Local symmetries and the what a gauge actually is.

The Changes in Our Lives - Local Symmetries

A local symmetry is a symmetry in the same sense as a global symmetry but the transformation has a dependence on x. For example, in the $e^{i\phi}$ case before if $\phi = \phi(x)$ then the transformation would form a local U(1) group. The process by which $\phi = \phi(x)$ is also known as gauging [13], and is the basis of local symmetries which are still a very active research areas as they provide the basis for the unifying theories such as QED (SU(2)) and QCD (SU(3)) and in the future, supergravity.

The process by which we gauge something is to replace a field, such as the EM-field, with a scalar and vector potential so we can express the Hamiltonian (and

Lagrangian) as functions of potentials. This process is best shown via the gauging of the EM-fields [14]. We define a vector potential $\mathbf{A}$ and a scalar potential ϕ in terms of the $\mathbf{E}$-field and the $\mathbf{B}$-field as [15],

$$\begin{aligned}\nabla \times \mathbf{A} &= \mathbf{B} \\ -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} &= \mathbf{E},\end{aligned} \tag{2}$$

and then these potentials are used in the Hamiltonian so the wave equation, or the matrix equation, can be solved for the eigenbases. These relations can be again shown to be invariant under the U(1) transformation we saw before, and so there is again charge conservation. The same can be done for parity and time and for CPT symmetry, and so they are all locally symmetric as well as globally symmetric. This was the result that lead to Yang-Mill's, which is based in group theory [12].

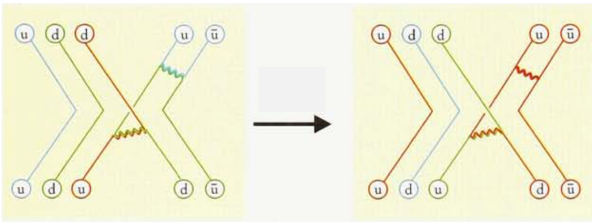

Figure 5: In this diagram the local system/process has changed but the overall result is exactly the same. This kind of symmetry is the underpinning of the sum over paths approach Feynman pioneered. That is, since there are an infinite number of ways to get to an end product, due to local symmetry, we must sum over all possibilities. Here we have an interaction described by $p + \pi^- \rightarrow n + \pi^0$ [2].

Another consequence of local symmetry, and Yang-Mill's theory, can be seen in figure 5; there are an infinite number of symmetric ways to get to the end product of $n + \pi^0$, and so due to the expansion postulate, we must sum over all possibilities. This summation procedure is Feynman's sum over all possible paths.

The most important feature consequence of local symmetries, and gauges, are the fact that they determine how particles and forces interact on a fundamental level [2]. The gauges themselves are the potentials in which the particles are moving, and those potentials are directly linked to the forces produced by a field. Eq 2 is an example of how the EM-field can be gauged so that the system can be described in terms of a family of potentials.

It is becoming more and more accepted by physicists that local symmetries are the true symmetries in nature and global symmetries are either just the leftovers of a broken local symmetry or the approximation to a slow-varying spatial symmetry [16]. This is more prevalent in quantum systems where the potential is slow-varying, such as a nuclear well, where the potential can be approximated to a system where global symmetries hold but an exact solution has no global symmetry.

Overall, local symmetries are the bread and butter of "theories of everything" today and gauging underpins the standard model, Quantum Field Theory and other fundamental theories in physics. So, whatever the outcome of the debate, gauging is still the most accurate process we have to describe our universe at the smallest scales.

Broken Symmetries and the Future

Symmetries dictate conditions on the laws of the universe and reveal to us much of the inner structure of the universe we live in, however, the majority of physical phenomena originate from the breaking of these symmetries. Modern theoretical physics is underpinned by the investigation of these breaking symmetries and they are inherent to theories with infinite degrees of freedom (namely field theories) [16].

The global symmetries we have analysed can usually be revealed via two mechanisms, the first of which is the standard method and is called the Wigner-Weyl mode [2]; in this method the laws of physics must be invariant under the transformation and there is a ground state which is symmetric (which is usually the vaccum). In quantum systems with finite degrees of freedom, this is standard. The other method (the Nambu-Goldstone mode) is another possibility in theories with infinite degrees of freedom. In this mode the ground state is asymmetric, and thus constitutes a breaking of symmetry.

This sounds strange but to clarify we'll consider a chunk of iron. If the ground state is symmetric, that is all of the domains point in different directions, then the iron has no magnetic properties. However, if all the domains point in the same direction, symmetry is broken and so we have magnetism, see figure 6. The majority of physical phenomena we observe occur due to a breaking of symmetry.

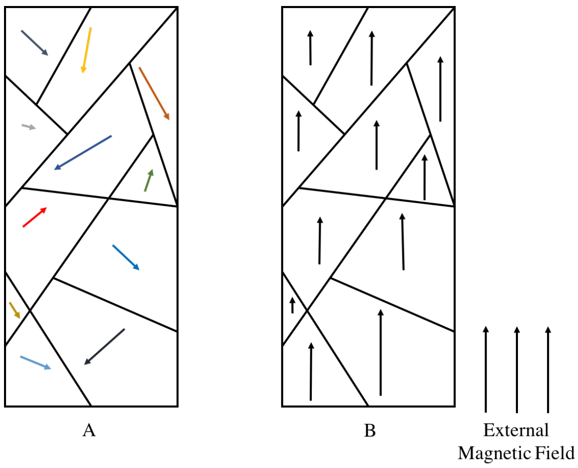

Figure 6: In A the iron domains are all randomly aligned so no magnetic field is actually present in the system, this is akin to a symmetric system. However, in B an external magnetic field is applied to the system so the magnetic domains align. This means the ground state is asymmetric and magnetism within the iron arises even when the magnetic field is removed.

Given that we have a method to find symmetries that are obvious, we now look for broken symmetries so we can find the underlying symmetry which has been broken.

This allows us to find symmetries which were previously disguised by the broken symmetry.

The exciting and most profound things about this area are the applications to fundamental physics and the search of a unified theory. Physicists now look for new local symmetries, and more intricate breaking of symmetries, in order to find the fundamental structure of the universe. We have seen that searching for these symmetries lead to QED and QCD, and so physicists hope that searching for more fundamental symmetries will allow us to incorporate gravity into this model. One of the most promising types of theories are so-called supersymmetric theories. In older theories symmetries occurred between fermions and bosons exclusively, but supersymmetry unifies the bosons and fermions into one model and one of the interesting consequences of this is the super-partners [2]. These partners are particles much like the fundamental particles but much more massive, as of yet they are unobserved but with the new LHC run it is hoped they may be observed experimentally. One of these supersymmetric theories is String-theory. It is by far the most credible unified theory we have which incorporates all of the fundamental particles and forces. It has all of the symmetries we know to be correct and some other, strange symmetries we have yet to understand.

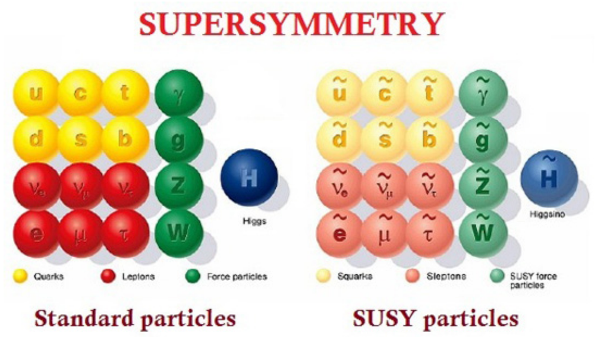

Figure 7: The standard particles have supersymmetric particles in the supersymmetric theories. We have yet to observe these particles but if we observe them it will be the first piece of concrete physical evidence we have for the existence of these theories [17].

Symmetries in nature and quantum mechanics have shown us a lot about the universe we previously did not understand; it has given a theoretical basis for many conservation laws, and revealed to us new theories to describe the universe. These symmetries are still being used in research today and underpin fundamental physics, from atomic spectra to the fundamental structure of the universe. Understanding how to find them and use them is instrumental for all physicists so we can continue to discover some of the most fundamental secrets of the universe.

My thanks go to Prof. Dimitri Vvedensky and Dr. Tim Evans, for answering questions on this topic, and to my friends who checked my article thoroughly.

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Article Plan
Callum Hunter
Area-Symmetry in Quantum Mechanics **Title-**TBC

Introduction: Introduce the importance of the topic by giving a little of the history around Wigner and atomic spectra [~450 words]

- Talk about symmetry in a very generic sense- reflections etc. (use images to introduce the concept)
- Mention how symmetries help see the regularities in the natural laws
- Briefly mention an example of a classical symmetry for a trivial example so the reader understands what we mean by a physics symmetry
- Finally bring in Wigner and his atomic spectra (cite paper) and mention a general outline for the article

Group Theory: Introduce what a group is in very basic terms and describe how they can be used to identify a symmetry (ie. By transformations forming groups) [~350 words]

- Define a group by its 4 main properties
- Give an example of a group (Likely to be a matrix group as this can be referred to later in the text for rotational invariances)
- Motivate the use of group since they are basis independent, and very abstract and so represent a far more general case
- Give extra places for the reader to look stuff up if (s)he wants to learn more about this area

Global Symmetries: Define and introduce the idea of global transformations, which lead to invariances and symmetries. Link this to important developments in physics [~1000 words]

- Define a global symmetry
- Define Noether's Theorem
- Bring in the three basic operations: Translation in time, space and rotation and link these to their conserved quantities
- Talk about how in QM we can sum the invariant equations to create a new solution
- Show the $e^{i\phi}\psi$conservation as an example of invariance and state what this leads to (charge invariance)
- Bring in the Parity and Time conservation laws (via the complex conjugate of the Wave function)
- Talk about how the above three lead to CPT symmetry and Yang-Mill's theory
- Mention that there are other global symmetries involving the rotational group defined before and also bring in that all global create group.

Local Symmetries: Define the local symmetries and what they mean. Discuss how they're useful in physics [~750 words]

- Define a Local symmetry and maybe discuss their inception during the 1960s for context
- Define and give the example of the EM gauging in QM which fixes the way the particles interact with the gauged fields
- Bring in how it forces the existence of special particles like Bosons etc.

Breaking Symmetries and the Future: Talk about how we believe all true symmetries are local and global ones are almost always broken (ongoing discussion) and talk about the future in terms of Quantum Gravity [~550 words]

- Bring in the breaking of symmetry (magnetism) and Gross' arguments for only local symmetry
- Finally bring in the idea of supersymmetry and quantum gravity
- Finish with a sentence highlighting the importance of this all

NB:

- There will often be images and diagrams explicitly showing the ideas discussed in this article since this area can be explained very visually as well as mathematically
- There will be equations but nothing that is above a 2nd year who has all the quantum taught in second year

This is an interesting but very technical subject. I hope you are able to explain the key concepts succinctly and at the appropriate level. There are a large number of subjects you are attempting to tackle so I hope you can tie them together into a coherent article that is not just a textbook.

Response:
The feedback highlights the issues that will occur a lot during the article, however by modelling my article off articles in high level magazines – such as "Physics Today" or "Physics World" – I hope to avoid these issues. Throughout writing the article I have comes across these problems but I hope by using clear explanation and succinct examples, the topics become easier to tackle.