Entanglement Entropy in Causal Set Theory

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Motivation

Causal Set Theory(CST) hopes to unify general relativity and quantum theory by imposing a minimum length scale on the universe, avoiding divergences. The core tenet of CST is that spacetime is a directed, acyclic graph, on which the nodes are spacetime events, and the edges, causality relations. The spacetime manifold then approximates this graph at large scales.

Our main focus was entanglement entropy(EE) within CST, which is believed to be a gateway to understanding quantum gravity. This is evidenced by the idea that black hole entropy may be EE[1]. Moreover, ideas based upon deriving the spacetime metric from quantum entanglement[2] may shed light on the interplay between the kinematics of general relativity and dynamics of quantum theory.

What are Causal Sets?

1. Locally Finite :
$$\forall x, y \in C, |I[x,y]| < \infty,$$
$$I[x,y] := J^+(x) \cap J^-(y)$$
2. Acyclic :
$$x \preceq y \,\&\, y \preceq x \Rightarrow x = y, \forall x, y \in C$$
3. Transitive :
$$x \preceq y \,\&\, y \preceq z \Rightarrow x \preceq z, \forall x, y, z \in C$$

$$J^+(x) = \{y \in C | x \preceq y\}$$
$$J^-(y) = \{x \in C | x \preceq y\}$$

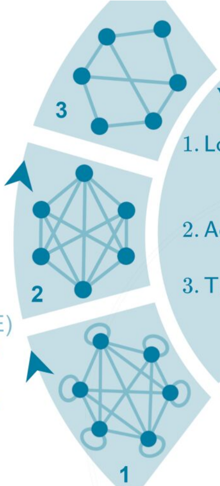

Fig 1: A locally finite graph(1), then made acyclic(2), which is further transitatively reduced to form the Hasse diagram of a causal set(3).

Entanglement Entropy

Von Neumann Entropy :
$$-\operatorname{Tr}(\rho \ln(\rho)) \qquad (1)$$
Entanglement Entropy :
$$-\operatorname{Tr}(\rho_A \ln(\rho_A)) \quad (2)$$
Complementarity :
For a bipartite system, such as fig.2 $S_A = S_B$ if ρ is pure.

ρ: density matrix
$$\rho_A := \operatorname{Tr}_B(\rho)$$

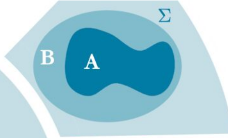

Fig 2 : A Cauchy hypersurface, Σ, subdivided into regions A and B.

Entanglement Entropy Background

EE is a component of a system's entropy that depends solely on the degree of its quantum entanglement with external regions.

To calculate the EE of region A in fig. 2, start with the full, pure, density matrix of quantum mechanics. Then, calculate the reduced density matrix of the region via the execution of a partial trace over the complement of region A, in this case, region B. Use that reduced density matrix in the von Neumann entropy formula equation (1), and the result is EE.

A result known in several fields is that the scaling of EE is proportional to boundary area. In relativistic quantum theory in 1+1d, it was found that this relationship obeys a logarithm, $log(x)/3 + b$, where x is the cutof[3].

Entanglement Entropy in Causal Sets

Causal Set Entropy :
$$S = \sum_{\lambda} \lambda \ln |\lambda|, \quad (3)$$
where λ satisfies
$$W\nu = i\lambda\Delta\nu. \qquad (4)$$

W : Wightman function
$i\Delta$: Pauli − Jordan operator

Framework

In CST, the notion of a Cauchy surface is ill-defined, and analogues simply do not contain the prerequisite information for a von Neumann entanglement entropy calculation to be performed. Due to this, Rafael Sorkin derived an expression for entropy based upon spacetime correlators, equation(3)[4]. This expression is covariant, and the eigenvalues lambda satisfy equation (4). These correlators carry information about the scalar field in the entire spacetime, and have a well-defined causal set analogue, due to causal set propagators being well understood.

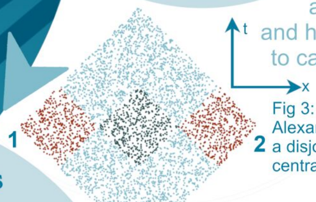

Fig 3: A causal set in the form of a Alexandrov interval, comprised of a disjoint region(red), and a nested central region(green).

Development

This entropy formula was implemented by Yasaman Yazdi to obtain the EE for a nested Alexandrov interval in the causal set, the region shaded dark green in fig. 3[5]. To obtain the desired spatial area law scaling, a truncation of the Pauli-Jordan operator was necessary.

We were tasked with extending this work to a configuration of disjoint intervals, the red shaded regions of fig. 3. To do so involved us finding the appropriate truncation in this new configuration, which required solving equation (5). We found the analytic solutions, equations (6) and (7), which turn out to enable the usage of the truncation scheme used by Yazdi. An example mode of the solution is shown in fig. 4. Now knowing the truncation scheme is shared, the treatment of multiple regions became clear, which is significant as there are few analytic results for the disjoint case.

Disjoint Regions

$$(5) \quad i\Delta f(X) := \int_M dV' i\Delta(X, X') f(X')$$

Solution 1 :
$$(6) \quad g_k(u,v) = \begin{cases} e^{-iku} + e^{-ikv} - 2cos(kl) & (u,v) \in 1\, or\, 2 \\ 0 & otherwise \end{cases}$$
$$k \in \mathcal{K} = \{k \in \mathbb{R} | tan(kl) = 2kl \,\&\, k \neq 0\}$$

Solution 2 :
$$(7) \quad f_k(u,v) = \begin{cases} e^{-iku} - e^{-ikv} & (u,v) \in 1\, or\, 2 \\ 0 & otherwise \end{cases}$$
$$k = \frac{n\pi}{l}, n = \pm 1, \pm 2, \ldots$$

fig 4: A plot of an example mode of the solutions to equation 5, an eigenfunction of the Pauli-Jordan operator. Produced via simulation of a causal set.

Results

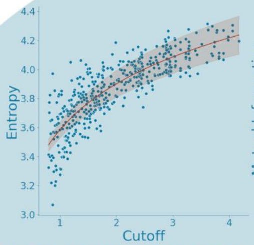

Fig 5: A plot of the entropy for a disjoint region against the cutoff. The setup is similar to fig. 3, but the red shaded regions are displaced towards each other.

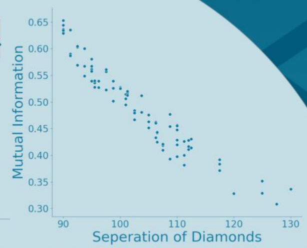

Fig 6: A plot of the mutual information against the seperation of the disjoint regions.

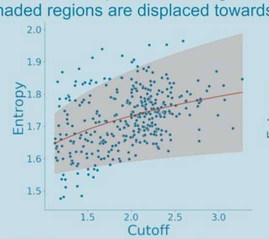

Fig 7: A plot of Entropy for a diamond located at the corner of an interval against the cutoff.

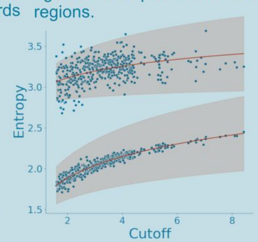

Fig 8: A plot of entropy for both the disjoint region(upper) and central region(lower) against cutoff.

Discussion

Having found the single diamond truncation applies in general, we investigated the EE scaling of various systems. We found that the mutual information of disjoint regions decays as their seperation increases, and that a single side diamond obeys a logarithmic scaling of coefficient 0.14±0.03. These are shown in fig. 6 and fig. 7, and both scalings agree with continuum results[3][6].

Fig. 5, the off-edge case has the same scaling as a single central diamond, a logarithm of coefficient 0.35±0.04. This is promising for reproducing complementarity, via investigating EE scaling as the diamonds approach the corners.

Finally, we examined complementarity. We found that the single diamond truncation scheme does not retrieve complementarity, seen in fig. 8. Modifying the trunction, certain configurations, hadmatching curves, though no general scheme was found.

The reproduction of continuum results gives us confidence in CST, and we have promising approaches to find complementarity. To confirm complementarity would surely be useful in furthering CST, and we hope to be able to do so.

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