Felix Swift Roberts

Title: Quantum Psuedo-Telepathy

Word Count: 2868

1. Magic Squares, Lightyears Apart

Let's play a game.

As far as games go, this one is fairly simple; you and I have been enlisted by some physicists to play it. We are to each draw a number of crosses in a division of a 3x3 grid, subject to conditions given by the researchers – you are to draw an even number of crosses in a row, and I am to draw an odd number of crosses in a column. The physicists choose which specific row or column we will each fill out, though we aren't allowed to know which specific division the other has been dealt – you know I'm filling out a column, but you don't know which one, and vice versa: obviously, we're not allowed to communicate during the game. We've each been supplied a 3x3 grid within which to draw our crosses: if our grids are compatible – that is, we can overlap them without contradiction – we win the game. Now, to make sure that we really don't communicate whilst the game is taking place, the researchers intend to separate us by several lightyears; I offer to take the burden of exile.

Once our draconian physicists have shipped me off to the nearest star-system, they turn to you and ask you to fill out the third row of the 3x3 grid with an even number of crosses. You can fill it out any way you want, so long as you meet those conditions – don't fill out any other part of the grid, and put a circle in any squares not filled with a cross.

Well, go on. Put this article down, grab a pen – someone around you is bound to have one! – and fill out the grid below. No hurry, I'll wait for you. Remember, an even number of crosses in the third row: it's even been demarcated. Make sure to put a circle in any unfilled square.

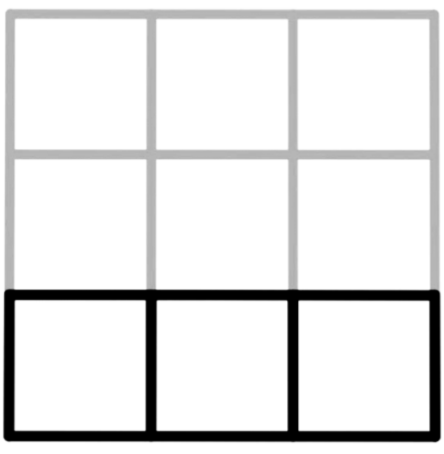

Fig. 1.1. Please fill in the third row of this grid with an even number of crosses. It would help quite a lot.

Done? Good. We're going to pretend that whilst you did yours I did mine, which you should find overleaf – I couldn't have you looking at my answers.

Upon reaching Alpha Centauri, I was instructed to fill the second column of my grid with an odd number of crosses – not having much else to do, I duly obliged; you can find my grid below.

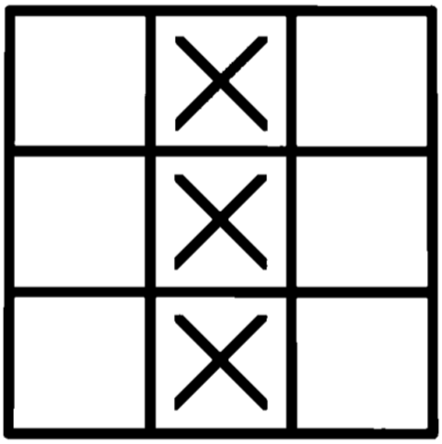

Fig. 1.2. My grid.

Compare the two grids: do the placements of the crosses match – that is to say, is there a cross in the 2nd box of the third row? If there is, congratulations! We've won the game! (If we didn't, don't take it personally; this one really is a matter of luck.) Though there's technically a 50% probability of winning in these conditions, other factors push that probability up slightly, so we most likely won[1].

Say the game was repeated, but this time we had the chance to speak before my exile beyond the solar system, though before the specific rows and columns were assigned. Being intelligent players, and knowing the rules of the game, could we come up with a strategy to win the game every time we played it? If we both memorised a consistent grid layout and copied out the specified column and row during the experiment, we could be sure of winning.

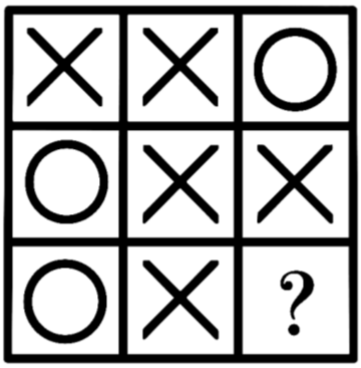

Fig. 1.3. When the parity rules are extended over the entire grid, a contradiction arises – the lower right square must contain a cross to meet the even row rule and a circle to meet the odd column rule.

When the entire 3x3 grid is filled out to meet the parity rules the issue becomes clear. There is no possible way to fill out the grid such that a contradiction doesn't arise (see Fig. 1.3). If we were

[1] There are four possible combinations you could have put in the grid, but "an even number" is usually read as "two"; excluding the possibility of zero crosses results in a $^2/_3$ chance of us winning.

to memorise Fig. 1.3 and play according to it, we could only expect to win eight times out of nine – if we were asked to fill out the third row and third column, we would clearly lose. Though in most other avenues a win rate of 8/9 would be exceptional, here it is simply not enough – there is no winning strategy.

Or, at least, that's the case classically.

2. Entanglement, Unknotted

"Do you really believe the moon doesn't exist when you are not looking at it?"
– Albert Einstein to Abraham Pais [2]

Einstein never reconciled himself with quantum mechanics.

To Einstein, the universe was deterministic and *knowable*: a sufficiently powerful intelligence could take the position of every particle in existence as its input and from it forecast a future as inexorable, as certain, as the winding-down swing of a pendulum. Of course, this calculation would be unfeasible, but the theoretical premise of it remains: with all the parameters of a system defined at an initial point, one should be able to know with certainty how that system will evolve. This is the central assumption of classical physics and, neglecting the special status often afforded human consciousness in these discussions, is how most of us tend to view the world.

The world of classical physics obeys both *locality* and *realism*. In a universe that obeys locality, the effects of an event can propagate at the speed of light, but no faster. If locality weren't true, faster-than-light signals could end up travelling backwards in time and wreak havoc on determinism. A realistic universe, on the other hand, is one where measurements only reveal intrinsic, already existing elements of reality. The position and momentum of a ball, for example, surely exist before they're measured.

And yet quantum mechanics seems to disagree with classical physics on the nature of the world. Electrons don't behave deterministically but probabilistically: the trajectories, the hard ineluctable track of parabolas, give way to diffuse probability clouds. Moreover, Neils Bohr, Einstein's great debate opponent upon the matter, argued that the uncertainty principle was no failure of human measurement, but rather reflected that it was only upon measurement that these properties came into existence!

It is hard to overstate how much Einstein objected to this view: the idea that a measurement "creates" what it measures is hard to accept intellectually, let alone philosophically. In an effort to prove that the position or momentum of a particle existed outside of measurement Einstein, alongside Boris Podolsky and Nathan Rosen, introduced the EPR paradox and, with it, entanglement. And yet, curiously, entanglement ended up being what decisively proved Einstein wrong on the matter – the world does not seem to obey local realism!

But what does all this have to do with magic squares played lightyears apart? In a local realistic world, there is no way to win the game played above every time. Using entanglement, however, it is possible to produce a winning strategy without communicating, even though players are lightyears apart; games that are unwinnable under local realism but that can be won using quantum effects are said to exhibit *pseudo-telepathy*.

3. The Machinery

So, what is entanglement? And how does entanglement lead to pseudo-telepathy?

We start with the qubit. This is the quantum analogue to the classical bit; whilst the bit exists in either one of two states, the general state of a qubit is as a superposition. A general qubit is thus written

$$|\psi\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle,$$

where α_u and α_d are the complex probability amplitudes associated with their respective states and are normalised as such; upon measurement, the qubit will collapse to either of the two states. For historical reasons, we choose the two z-axis spin states, up, $|u\rangle$, and down, $|d\rangle$ as our basis [2, 3]. We expect the two states to be orthogonal to each other - we mathematically represent them as follows:

$$|u\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |d\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

A qubit therefore exists in a 2-dimensional space.[2]

What about a system with multiple qubits? We describe a system of two uncorrelated qubits by taking their *tensor product*. The tensor product of basis states $|u_1\rangle$ and $|u_2\rangle$ is written $|u_1u_2\rangle$, and is an entirely new basis state: the tensor product of two *n*-dimensional spaces produces a new 2*n*-dimensional space. The tensor product for two 2x2 matrices is seen below.

$$A \otimes B = \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \otimes \begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix} a_{11}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} & a_{12}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} \\ a_{21}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix} & a_{22}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}.$$

The tensor product of two up states is

$$|u\rangle \otimes |u\rangle = |uu\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},$$

and the tensor product of two qubits

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_u\beta_u|u_1u_2\rangle + \alpha_u\beta_d|u_1d_2\rangle + \alpha_d\beta_u|d_1u_2\rangle + \alpha_d\beta_d|d_1d_2\rangle.$$

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[2] The observant reader might notice an issue – if a qubit exists in a 2-dimensional space, it will be specified by 2 degrees of freedom, and yet the complex probability amplitudes give 4 degrees! The solution is that the normalisation condition eliminates a degree of freedom, and another degree of freedom is represented in an arbitrary phase factor that can also be eliminated; we are left with two degrees necessary to specify the qubit. Note too that although $|u\rangle$ and $|d\rangle$ are mathematically orthogonal, that is no indication of the spatial relation – we take them to be antiparallel physically.

The two 2-dimensional qubit spaces produce a 4-dimensional two-qubit space. Rather than subscripting each element within the ket, we note that the first qubit's contribution, so to speak, appears first – the subscript is, from now, implied in the position within the ket. Just to reiterate: although the basis state $|ud\rangle$ is formed from both $|u\rangle$ and $|d\rangle$, it is a distinct basis state in a different space. If we measure the state of $|\psi_1\rangle$ and $|\psi_2\rangle$ as $|ud\rangle$ in our two-qubit space, $|\psi_1\rangle$ is in state $|u\rangle$ and $|\psi_2\rangle$ in state $|d\rangle$.

More systems exist in this 4-dimensional space than are represented by tensor products: for example, a system of the form

$$|\psi\rangle = \frac{|du\rangle - |ud\rangle}{\sqrt{2}}$$

cannot be decomposed into two separate uncorrelated qubits. This is an *entangled* state – in fact, it is a maximally entangled state, or a *bell state.* Notice that if we take a measurement of one of the constituent qubit's spins in the z-axis and find it to be in the state $|u\rangle$, the other is clearly in the state $|d\rangle$; this property extends to any possible direction in which a qubit's spin might be measured – if we measure one qubit as $|u\rangle$ along a specific direction we will, with certainty, find the other to be $|d\rangle$ along the same direction. It is this that Einstein used to strike at non-realism; if we can deduce the state of a spin without directly measuring it, surely it exists before measurement?

Many physicists weren't persuaded by Einstein's argument, and three decades after the EPR paradox was introduced John Bell was to prove that entanglement entailed statistics that denied local realism – in the interests of space, they will not be covered; we will content ourselves with the fact that a successful demonstration of quantum pseudo-telepathy would be a demonstration of Bell's theorem [4].

Finally, now that we're sufficiently versed in qubits, we must raise the question of what we may do with them: the three fundamental operators that concern us are the Pauli matrices. They are Hermitian, and correspond to the spin components in the *z*-, *y*-, and *x*-axes respectively:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Acting with σ_z upon our basis vectors gives

$$\sigma_z|u\rangle = |u\rangle$$

and

$$\sigma_z|d\rangle = -|d\rangle;$$

both are eigenvectors, with eigenvalues +1 and -1 respectively. The other Pauli matrices can operate on the *z*-axis basis states (the results are depicted in table 3.1), and the tensor products of the Pauli matrices with themselves and the identity matrix *I* can operate on two-qubit systems.

Operator	$\|u\rangle$	$\|d\rangle$
σ_z	$\|u\rangle$	$-\|d\rangle$
σ_y	$i\|d\rangle$	$-i\|u\rangle$
σ_x	$\|d\rangle$	$\|u\rangle$

Table 3.1. The results of acting open $|u\rangle$ and $|d\rangle$ with the Pauli matrices.

4. *Winning the Game*

Being experienced participants, the physicists call us back one last time to fill in the squares. However, before we start the game we share two pairs of entangled particles (4 qubits), each of us taking one particle from each pair, memorise the magic square shown in fig. 4.1, and we each surreptitiously pocket a spin-measurer. We imagine this device to be able to measure any of the observables shown in the magic square as instructed and to have ouputs +1 and -1.

Once I am safely away in orbit around Alpha Centauri with my two particles and detector the physicists tell you to fill the first row with an even number of crosses. Roughly simultaneously[3], I am told to fill the first column with an odd number of crosses.

$I \otimes \sigma_z$	$\sigma_z \otimes I$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes I$	$I \otimes \sigma_z$	$\sigma_x \otimes \sigma_x$
$-\sigma_x \otimes \sigma_z$	$-\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Fig. 4.1. The magic square of observables that wins the game. Adapted from [4].

The state of the two entangled pairs can be written as

$$|\psi\rangle = \frac{|dd\rangle_{1,2} + |uu\rangle_{1,2}}{\sqrt{2}} \otimes \frac{|dd\rangle_{3,4} + |uu\rangle_{3,4}}{\sqrt{2}}$$

$$= \frac{1}{2}(|dddd\rangle + |dduu\rangle + |uudd\rangle + |uuuu\rangle)$$

$$= \frac{1}{2}(|dd\rangle_{1,3}|dd\rangle_{2,4} + |du\rangle_{1,3}|du\rangle_{2,4} + |ud\rangle_{1,3}|ud\rangle_{2,4} + |uu\rangle_{1,3}|uu\rangle_{2,4})$$

Given that we both take a particle from each of these pairs, we can both write the state of our two particles as

$$|\psi\rangle = \frac{1}{2}(|dd\rangle + |du\rangle + |ud\rangle + |uu\rangle).$$

Upon measurement, the wavefunction will collapse with equal probability to any of these spin eigenstates and will give away the state of the other's particles. We determine what we write based

3 Simultaneously enough that there isn't time for a signal from Earth to reach me, which gives me roughly 4 years to be told to fill out the grid.

upon the results of the measurements; for a measurement of a given observable of our particles, if we obtain +1, we draw a nought, and for -1 we draw a cross.

You run your two particles through the spin-measurer, measuring all the observables in the first row – they all commute, so there are no issues regarding order. A sample calculation for the first element is given:

$$I \otimes \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$I \otimes \sigma_z |\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

$$= \frac{1}{2}\left[-\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right]$$

Clearly, we obtain a result of -1 from $|dd\rangle$ and $|ud\rangle$ and an eigenvalue of +1 from $|uu\rangle$ and $|du\rangle$. The results of the rest of the first row are given in the table below:

	Measured Value			
Observable	$\|dd\rangle$	$\|du\rangle$	$\|ud\rangle$	$\|uu\rangle$
$I \otimes \sigma_z$	-1	+1	-1	+1
$\sigma_z \otimes I$	-1	-1	+1	+1
$\sigma_z \otimes \sigma_z$	+1	-1	-1	+1

Table 4.1. The results of acting open the 4 two-qubit basis states with the first row operators.

Notice that the parity rules are already visible – we see an even number of crosses in each column! Only two measurements need to be made – the parity condition allows the last result to be figured out.

Say you observe the result (-1,-1,+1). Clearly, the wavefunction has collapsed to the state $|dd\rangle$. But where does that leave my measurements? If we assume you had taken your results and collapsed the wavefunction before me, a quick calculation shows that $|uu\rangle$ is not an eigenstate of the other operators in the first column. However, $|uu\rangle$ can be written as an equally-weighted superposition of eigenstates of the first row:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

	Measured Value	
Observable	$\|dd\rangle + \|ud\rangle$	$\|dd\rangle - \|ud\rangle$
$I \otimes \sigma_z$	-1	-1
$\sigma_x \otimes I$	1	-1
$-\sigma_x \otimes \sigma_z$	1	-1

Table 4.2. The results of acting open the two components of the $|dd\rangle$ superposition with the first column operators.

Running either through the first column's operators yields the results shown in table 4.2.

The parity laws for the columns are clearly obeyed – not only is there no discrepancy between our shared element but all conditions are met. Similarly, if I measured the wavefunction first and collapsed it to one of the above states, it would exist in a superposition of viable states for your first row – the strategy holds even if we move such that, by relativity, we both collapse the wavefunction first!

No information has actually been sent between us: I never learn your row, and you never learn my column; and yet, using entanglement, we obtain a result as if we'd had. So, how has this come about? The first innovation is that the collapse of the entangled wavefunction ensures our results will always be consistent: without communication, we can nonetheless be completely confident as to the state of the other's particles upon making our measurements. The other innovation involves the magic square itself. Though the parity laws on the 3x3 grid cannot be fulfilled with numbers, they can be met with matrices – the product of each row's observables is the identity matrix, and the product of each column's observables the negative of the identity matrix. If we restrict our grid elements to +1 and -1, it is clear this is simply our parity rule.

5. Endgame

The magic grid is not the only pseudo-telepathic game, but it is the simplest and, I think, the most compelling. Though you will hopefully have appreciated pseudo-telepathy for what it is, you might wonder what exactly it is useful for – a not unreasonable question. Firstly, as a simple demonstration of quantum "weirdness", it is a remarkably simple introduction to the consequences of a world that does not follow local realism – the fundamental result can be appreciated without knowing the mathematics hidden behind it. Secondly, though Bell's theorem has been demonstrated experimentally, loopholes that fail to completely rule out local realism continue to plague experimental demonstrations; a demonstration of a pseudo-telepathic game could be the first loophole-free verification of Bell's theorem. And, at any rate, in the far future, if we throw ourselves out amongst the stars, pseudo-telepathic games might give us something to do with those back home during the long dark days around Alpha Centauri. Trust me; I'd know.

6. Bibliography

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