From Manifolds to Margherita - The Differential Geometry of Pizza

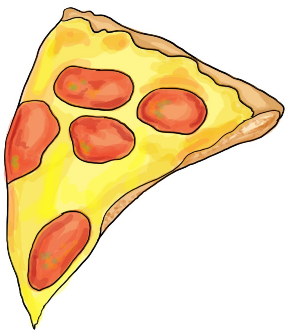
Word Count: 2989

From Manifolds to Margherita - The Differential Geometry of Pizza

by LUCÍA CUERVO VALOR

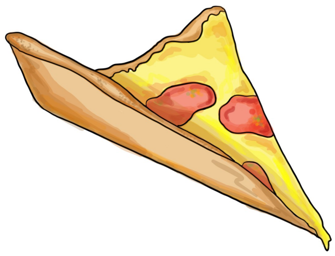
"In his youth Albert Einstein spent a year loafing aimlessly. You don't get anywhere by not 'wasting' time", explains Carlo Rovelli in his sensation *Seven Brief Lessons of Physics*, "It is thus that serious scientists are made" [1].

No physicist can accomplish truly productive work without relaxing from time to time; physicists need, too, an occasional Friday night with friends and, no question, some pizza. However, being the physicists that they are, even on the Fridays while eating pizza, they cannot help but wonder the mathematically logical way for them to eat the (often many) slices.



'Why does my slice bend down if I grab it flat by the crust, but not if I care to bend it into a U-shape? Is it following some rule? What sort of strange geometrical object is in my hands?'.

As it turns out, geometry poses restrictions on the way in which shapes can change. This can be understood through the study of curvature for curves and surfaces. Not only can this become important in countless examples of the every-day life, but topology presents intricacies that are key in many contemporary scientific challenges.



1. Curves - What are those?

1.1 Describing curves

In order to be able to understand the topological properties of surfaces, it merits to start from the more basic concept: curves.

A mathematical description of a curve in $\mathbb{R}^2$, named a *plane curve*, can be given in terms of its Cartesian equation [2]

$$f(x, y) = c \,, \tag{1}$$

where f is a function of x and y, and c is a constant. We can describe curves in $\mathbb{R}^3$, or *space curves*, in this way, too, by using two Cartesian equations [2]

$$\begin{aligned} f_1(x, y, z) &= c_1, \\ f_2(x, y, z) &= c_2. \end{aligned} \tag{2}$$

However, it's often easier to use a different approach to describe curves. This consists of imagining it as a single point that moves along a 'path'. Say γ is the position of the point at time t. Then the function $\gamma(t)$ describes the curve. We call these *parametrized curves*, from the idea that the variables in its Cartesian equation are re-written in terms of a scalar parameter, t. Note that γ is a point in space, so it takes vector values [2].

The reason why this is more useful is that we can apply concepts to do with motion, which are normally intuitive to us, to the problem. We can use vectors to interpret derivatives, and algebraic concepts become geometrical.

1.2 Unit-speed parametrization

For a parametrised curve γ, we call $\dot{\gamma}$, its first derivative, the *tangent vector*. Since we have chosen for the curve to be a moving point, $\dot{\gamma}$ will be equivalent to the point's velocity vector, and $|\dot{\gamma}(t)|$ to its speed.

There are (infinitely) many equivalent parametrizations for each curve. We say that a parametrized curve is *unit-speed* if $\dot{\gamma}$ is a unit vector $\forall t$ (in the interval in which the curve is defined) [2].

Note that the curve being unit-speed allows for us to interpret its second derivative, $\ddot{\gamma}$, as a vector perpendicular to the tangent vector, hence normal to the curve. This can be understood if we think of $\ddot{\gamma}$ as the acceleration vector. Since the speed is always unitary, we know $\ddot{\gamma}$ never changes γ's modulus, but only its direction. This means $\ddot{\gamma}$ must be perpendicular to $\dot{\gamma}$.

2. Curvature of a Plane Curve

2.1 The mathematics

From the word itself, *curvature* becomes an intuitive concept: it tells us, indeed, just how much a curve curves. We will say that curvature, κ, measures "the extent to which a curve is not contained in a straight line" [2]; that is, to measure κ at point $\gamma(t)$ of the curve, we look at its deviation from the tangent line at t, and this deviation will be proportional to the curvature.

Let us define a *normal unit vector*, $\mathbf{n}$, as a unit vector perpendicular to the tangent vector $\dot{\gamma}$. Figure 1 shows how one can use $\mathbf{n}$ to find that the deviation of the curve from its tangent line at $\gamma(t)$, as t changes to Δt, will be equal to $(\gamma(t+\Delta t)-\gamma(t))\cdot\mathbf{n}$.

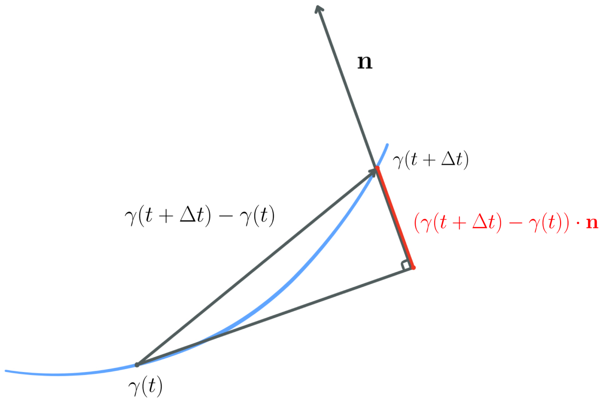


Fig. 1 - The deviation, shown in red, of a curve, in blue, from its tangent line at $\gamma(t)$, obtained using a normal unit vector indicated as $\mathbf{n}$. Recreated from [2].

Working from a Taylor expansion of $\gamma(t+\Delta t)$ we can find that

$$\begin{aligned}\gamma(t+\Delta t)-\gamma(t) &= \dot{\gamma}(t)\Delta t \\ &+\frac{1}{2}\ddot{\gamma}(t)(\Delta t)^2 + \mathit{HOTs}.\end{aligned} \tag{3}$$

HOT stands for *'Higher order term'*. When multiplying this expression by $\mathbf{n}$, we can notice that $\dot{\gamma}\cdot\mathbf{n}=0$, so the deviation of γ from its tangent line is

$$\frac{1}{2}\ddot{\gamma}(t)\cdot\mathbf{n}(\Delta t)^2 + \mathit{HOTs}. \tag{4}$$

Since $\ddot{\gamma}$ is perpendicular to $\dot{\gamma}$, it will be parallel to $\mathbf{n}$. Hence, the magnitude of the deviation, neglecting *HOTs*, will be

$$\frac{1}{2}|\ddot{\gamma}(t)|(\Delta t)^2. \tag{5}$$

We discussed that κ must be proportional to this, allowing for the definition of curvature to become [2]

$$|\kappa| = |\ddot{\gamma}(t)|. \tag{6}$$

Notice that a more complete definition requires for a distinction between positive and negative curvatures. We define the *signed curvature*, k_s, with the same approach, only making a choice that the normal vector used to derive the expression is, by convention, obtained by rotating $\dot{\gamma}$ anticlockwise by $\pi/2$ [2]. This will still be parallel to $\ddot{\gamma}$, and so there exists the scalar k_s such that

$$\ddot{\gamma} = \kappa_s\mathbf{n}. \tag{7}$$

Figure 2 shows an example of curves with positive and negative curvatures.

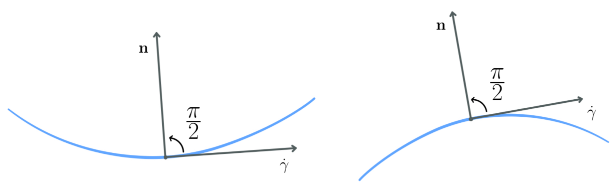


Fig. 2 - On the left, a curve with positive curvature; on the right, one with negative curvature. Normal vector chosen as per convention. Inspired from [2].

Thinking of $\ddot{\gamma}$ as the acceleration, the rate of change of direction of γ's motion, it makes sense that it is directly related to the curvature.

Briefly note that these definitions only hold for unit-speed parametrized curves. Curvature can also be defined for regular curves, but that derivation is not contained in this article.

2.2 Hidden circles.

Consider the unit-speed parametrization $\gamma(t)$ of a circle of radius R and centered around (x_0, y_0) [2]

$$\gamma(t) = \left(x_0 + Rcos\frac{t}{R}\;,\; y_0 + Rsin\frac{t}{R}\right). \tag{8}$$

From this, its curvature, obtained per Equation 6, is

$$|\ddot{\gamma}(t)| = \sqrt{\left(-\frac{1}{R}cos\frac{t}{R}\right)^2 + \left(-\frac{1}{R}sin\frac{t}{R}\right)^2} = \frac{1}{R}. \tag{9}$$

The curvature of a circle is the reciprocal of its radius, in accordance to sensible expectations; a smaller circle will curve more rapidly than a big one.

With this in mind, it's easy to see that a plane curve's curvature can be visualised by finding a circle that perfectly 'hugs' the curve (mathematically equivalent to making their second derivatives match, i.e. making their curvatures match!). An example of this is shown in Figure 3.

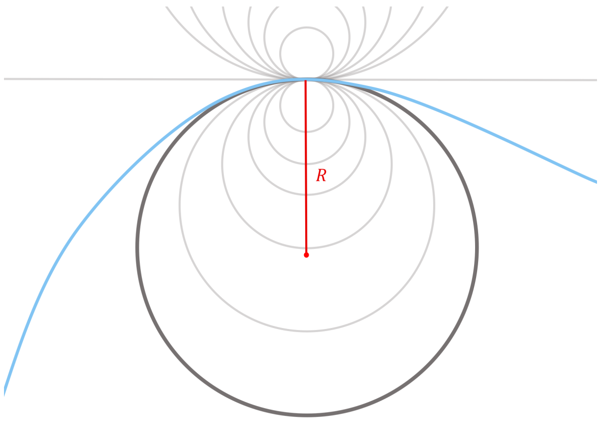


Fig. 3 - A curve is shown in blue. In dark gray is its properly 'hugging' circle and in light gray, many circles that don't fit appropriately. In red, R is the radius of the fitting circle.

Of course, this isn't necessarily always helpful when *computing* the curvature, but it is very useful for visualising it. Moreover, it will become an intuitive tool later on, when looking at surfaces.

3. Curvature of a Space Curve

3.1 Torsion is the new curvature

Curvature isn't sufficient to uniquely describe a space curve, because in three dimensions many curves may have the same curvature without it being possible to relate them by an isometry of $\mathbb{R}^3$ (a distance-preserving transformation) [2]. One visual insight to why this is true might be to think that the curve could be 'rotating over itself'. If, per say, the curve had a diameter and we marked a dot on it, it's possible that this dot could move around the diameter as if the curve was somehow being twisted - as represented in Figure 4. This idea arises when we realise that the curve is now in three-dimensional space and so now it has the new possibility to turn over itself.

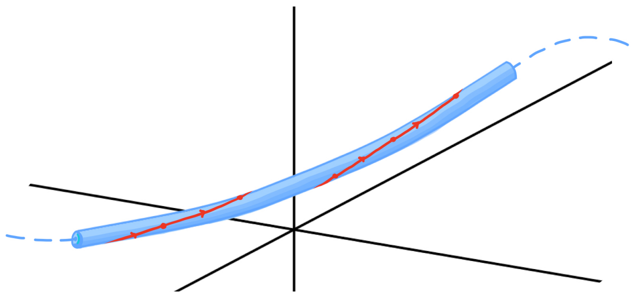


Fig. 4 - Explanatory diagram to show a space curve twisting.

Of course, the curve doesn't actually have a diameter, but the idea that it can twist remains. Another way to picture this could be to see the curve as the path of an airplane, like in Figure 5.

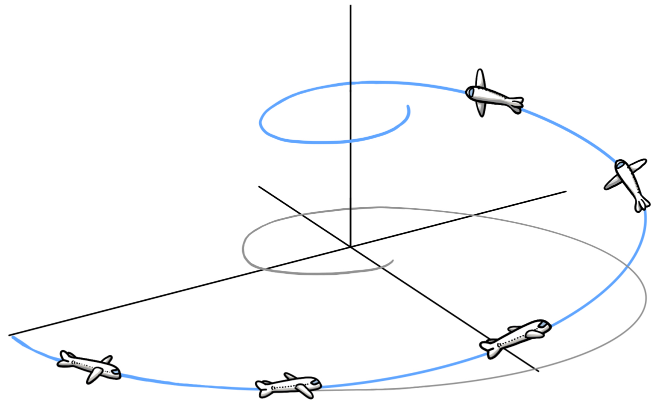


Fig. 5 - Explanatory diagram showing a different visual insight for a space curve twisting. The aim is to show that the curve doesn't need a diameter to twist.

To fully describe the curve we must now somehow account for this 'additional curvature', denoted the *torsion*, τ. The approach to finding τ will be similar to the one seen for curvature previously, only considering a new, different normal unit vector. This vector must be perpendicular to both $\dot{\gamma}$ and $\mathbf{n}$, and is therefore named the *binormal vector*, $\mathbf{b}$, and defined such that [2]

$$\mathbf{b} = \dot{\gamma} \times \mathbf{n}. \tag{10}$$

Figure 6 shows an example of this.

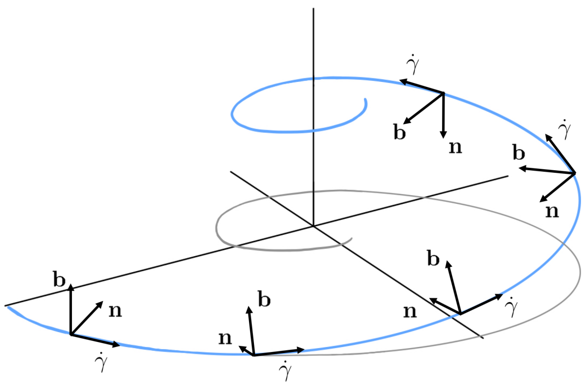


Fig. 6 - Vector analysis at different points on a space curve.

We can look at $\dot{\mathbf{b}}$ which, using the product rule, is

$$\dot{\mathbf{b}} = \ddot{\gamma} \times \mathbf{n} + \dot{\gamma} \times \dot{\mathbf{n}} = \dot{\gamma} \times \dot{\mathbf{n}}, \tag{11}$$

because $\ddot{\gamma} \times \mathbf{n} = \kappa\mathbf{n} \times \mathbf{n} = 0$. Note that $\dot{\mathbf{b}}$ is perpendicular to $\dot{\gamma}$, and also to $\mathbf{b}$, so it must be parallel to $\mathbf{n}$. Then there is a scalar, say τ, such that

$$\dot{\mathbf{b}} = -\tau\mathbf{n}. \tag{12}$$

This closely resembles the description of κ. The negative sign is added conventionally [2].

Note, again, that this applies for unit-speed parametrized curves only. The torsion definition for regular curves is not contained here.

4. Curvature of a Surface

4.1 The fancier space curves

A surface is, like a space curve, an object of $\mathbb{R}^3$. It "looks like a piece of $\mathbb{R}^2$ in the vicinity of any given point" [2] - if a curve is a curved line, a surface is a curved plane.

Having understood the ways in which space curves curve helps us understand curvature for surfaces, although the final idea is not the same.

Suppose we had a space curve laying on a surface, even have it casually twist in the same way that the surface does, and analyse $\mathbf{b}$, $\dot{\gamma}$ and $\mathbf{n}$ for it. See Figure 7.

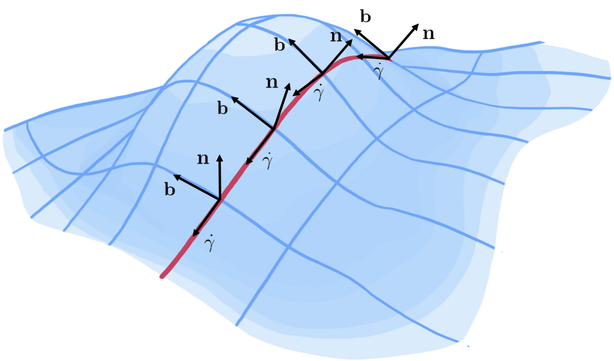


Fig. 7 - Vector analysis for a space curve that is coincident with a surface.

The need to account for $\mathbf{b}$ and torsion applies for surfaces, with one key difference. Notice that $\mathbf{b}$ is not normal to the surface, as it was to the curve, but is, in fact, tangent to it. Notice also that we can find infinitely many tangent vectors like this, which could even create a *tangent plane*. The curvature of a surface measures how much the surface is not contained in the tangent plane [2].

Even if it's still not apparent how we calculate the final curvature, it should be instinctive by analogy with space curves that, to get a value for the curvature of a space shape, we could look at 2D curvatures of two different plane 'projections' or sections of it. It might even be logical, by now, to expect that these two sections might better be perpendicular.

4.2 Principal curvatures

Let us now think of the *normal plane*, one that contains $\mathbf{n}$ and a tangent vector [3]. Since there are infinite tangent vectors, there are also infinite normal planes.

Each of these planes will intersect the surface and create a cross-section, named *normal section*. A normal section is a plane curve, and so it has a curvature κ. We could construct infinitely many normal sections, but we will be interested in two particular ones; the ones with maximum and minimum curvatures, κ_{max} and κ_{min}. These curvatures are called the *principal curvatures* [3].

Funnily enough, principal curvatures always correspond to planes that are perpendicular to each other (unless $\kappa_{max} = \kappa_{min}$ in which case κ is the same for all normal sections and it doesn't really matter).

It should be acknowledged that there exists a more formal definition for the principal curvatures, based upon the introduction of new concepts like the first and second fundamental forms and the Weingarten map [2]. However, these are subtleties that become important in other types of problems and are not necessary for this study.

4.3 We prefer circles

To find out κ_{max} and κ_{min}, we can look at their appropriately fitting circles (recall section 2.2). Hence, for surfaces, the curvature at a point can be calculated by finding *two* circles, the biggest and smallest possible, which 'hug' the surface. An example is shown in Figure 8.

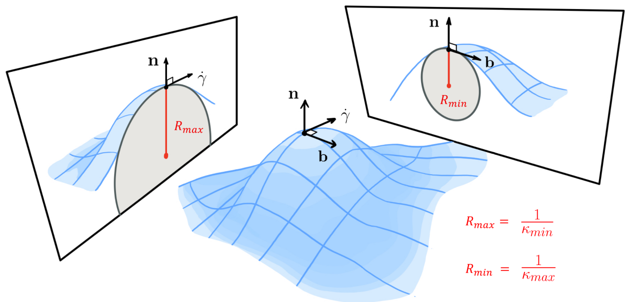


Fig. 8 - Analysis of the principal curvatures for a surface using circles. Recreated from [4].

This idea is useful because it makes curvature visually intuitive.

The problem now is that, even if we can find these two circles (i.e. the principal curvatures), we still haven't discovered a way to calculate a final, single, relevant value for the curvature of a surface. Here is where the famously called 'prince of mathematics'[1] comes in; as often in mathematics, Carl Friedrich Gauss gave us the key to the solution.

4.4 Gauss, the knight in shining armour

We define the *Gaussian curvature*, K, as the product of the principal curvatures

$$K = \kappa_{max} \cdot \kappa_{min} \tag{13}$$

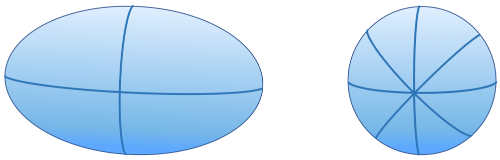
and the *mean curvature*, H, as the average of them [2]

$$H = \frac{1}{2}(\kappa_{max} + \kappa_{min}). \tag{14}$$

These together have incredible power in describing surfaces.

Gaussian curvature tells us information about the type of point upon examination. In fact, it helps us classify points [5]:

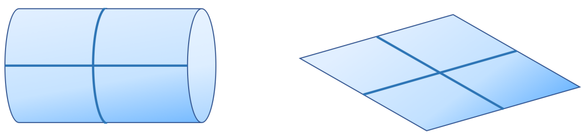
- *Elliptical points* exist where both principal curvatures have the same sign; that is, $K > 0$.
  - Elliptical points can be *umbilic*, when $\kappa_{max} = \kappa_{min}$. If $\kappa_{max} = \kappa_{min} \neq 0$, the point is said to be *spherical*.



Elliptical Spherical

Fig. 9 - Examples of elliptical and spherical surfaces. Recreated from [5].

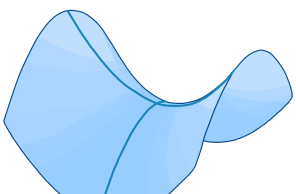
- *Parabolic points* exist where one of the principal curvatures is zero, and so $K = 0$, too.
  - If $\kappa_{max} = \kappa_{min} = 0$, the point is *planar*, or a *flat umbilic*. This is the only umbilic that is not elliptical.



Parabolic Planar

Fig. 10 - Examples of parabolic and planar surfaces. Recreated from [5].

- *Hyperbolic points* exist where the principal curvatures have opposite signs, so $K < 0$.



Hyperbolic

Fig. 11 - Example of a hyperbolic surface. Recreated from [5].

5. Theorema Egregium

5.1 The 'remarkable theorem'

Not only did Gauss define K, but he also explained one of its most interesting utilities, which he, indeed, named *Theorema Egregium*, latin for 'remarkable theorem' [2]:

"*The Gaussian curvature of a surface is preserved by local isometries*".

This means that if we bend a surface without doing anything tricky like stretching, shrinking, tearing or wrinkling it, its K must remain the same. A proof of this can be found in W. Ballman (2018) [6], by considering more formal definitions of κ_{max} and κ_{min}.

The important thing about this is not just that the curvature itself won't change - we don't often care about the exact value of K. What we do care about is what this implies about the surface; that is, there exist constraints to the way in which it can naturally change shape. If it is restricted to a certain K, there are alterations that will be allowed, and others that will be forbidden.

See Figure 12 as an example. You can easily turn a piece of flat paper into a cylinder without wrinkling it, but not into a sphere. Think about having to wrap something spherical - you just cannot get it right [7].

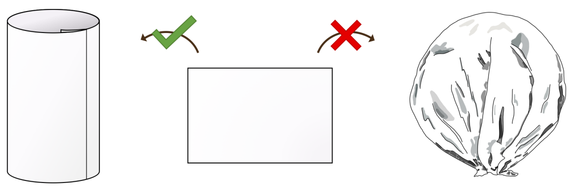


Fig. 12 - Explanatory diagram showing how a sheet of paper can be rolled into a cylinder, but will not be able to naturally become a sphere.

Alternatively, you can't create a flat surface from a curved one either, in the same way that you can't make a mandarin's skin flat without tearing it. This is actually the reason that we cannot create an accurate map of the Earth; you either get its angles right, or its areas, but never both at the same time [2]. This is because, per Theorema Egregium, there is no direct isometry from one to the other, and so distances aren't preserved.

Following from this, surfaces are usually classified into two types. The first is *developable surfaces*, with $K = 0$, which essentially are Euclidean shapes (this includes shapes like planes, cylinders, cones, and more). The second type are called *intrinsically curved surfaces*, with $K \neq 0$, which are non-Euclidean shapes [8].

5.2 Flat means flat

I know, I know - you're probably wondering what any of this has to do with pizza. Theorema Egregium is actually the self-contained answer to our original question: why is it that a slice of pizza doesn't bend down as usually when we hold its crust like a U? To understand this, see Figure 13.

The pizza was completely flat when it was on the plate, so its Gaussian curvature is, for sure, zero. When we hold the crust flat, we're making one of the principal curvatures zero, and giving the slice permission to curve

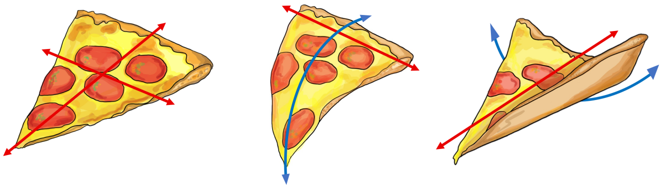


Fig. 13 - Diagram shows a slice of pizza bent in three different ways. Principal curvatures for each of them are shown with arrows. Red arrows indicate principal curvatures that are equal to zero.

in another direction. But if, instead, we take control and curve the crust, we ensure for there to be another normal section with $\kappa = 0$ - we forbid it to bend in the downwards direction.

This concept is actually incredibly powerful for many real-life applications. Take, for example, the structure of cardboard - this is represented in Figure 14. Because the wrinkles have curvature in one direction, this forbids them to bend in the other direction, giving the structure an incredible strength, even being made of really thin paper. This type of structure is called a 'corrugated material', and its design is common in architecture, too [7].

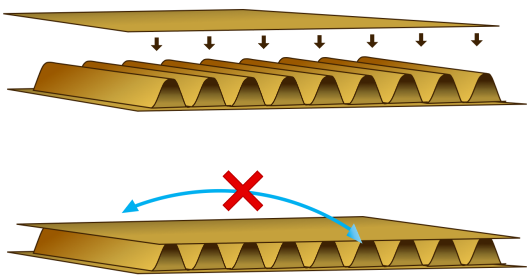


Fig. 14 - Above, a diagram showing the inner structure of cardboard. Below, a diagram showing the utility of this structure; that it provides a strong stiffness in a certain direction. Inspired from [7].

Actually, the utility of Theorema Egregium extends further than to just flat surfaces. Notice that, as hard as it is to crash developed surfaces in certain directions, it's even harder to crash an intrinsically curved surface. When $K = 0$, all the normal sections can curve in any way as long as just one of them stays flat. But for curved surfaces, K will be a very specific number that only a smaller set of principal curvatures can satisfy, and so the surface will be much more resistant to change, especially if the aim is to make it completely flat. This idea is also used in architecture! [7]

Further interesting examples of uses of flat and curved shapes for creation or even of the ways in which these appear in nature can be read in Bhatia A. (2014) [7].

5.3 What about reversing surfaces?

Imagine that we made a bowl-shaped pizza. Isn't it allowed for a slice to bend down like in Figure 16, as long as it bends down in both directions and the final curvature is the same?

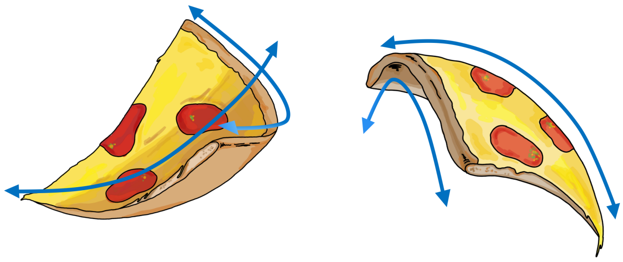


Fig. 15 - On the left an example of a slice of pizza, if this pizza was prepared in the shape of a bowl. On the right, a reversed version of this slice, with principle curvatures equal but opposite to the previous slice.

The answer is no, because, although intuitively it may seem like the surfaces are similar, the curved-up slice bending down would require for it to be flat at some point along the way, which is forbidden! This is the same as trying to turn a sphere inside-out. Of course, the final shapes would be the same and one might think it's possible to go from one to another - but it isn't, because we can't carry out the process of reversing it without stretching the surface. So there's another way to eat your pizza right if one has time to kill at home.

5.4 Could use a man on the inside

Leaving pizza aside, Theorema Egregium is a very important statement because it implies that Gaussian curvature is an *intrinsic* property [2]. It is something inherent to the surface, because of how it is and not because of its circumstantial situation or state. It tells us information about the *nature* of the surface.

This is different for mean curvature [8]. Mean curvature is useful if we want to know about the *condition* of a surface, because it is sensitive to the way it's bent or modified.

One of the most phenomenal implications of K being an intrinsic property is that it can be measured from within the surface [2]. For example, if an ant was to live on, per say, a cylinder, it wouldn't be able to tell the difference between this and a plane, because both have $K = 0$. The ant might as well think that it's living on a flat surface, because it is, in fact, living on a flat surface!

On the other hand, if the ant was living on a sphere, it could recognize through some calculation (this is a very clever ant) that the curvature of the surface where it lives is positive. Take the Earth as an example. If someone were to travel from the North Pole to the Equator, turn 90° to either side and walk a quarter of the Earth's radius, only to turn 90° to the same direction as before and walk some more, they would find that they end right back where they started. These were three straight lines, so the path must have been a triangle. But it 'swiped' more than 180°, which is impossible within a triangle. How can this be? The answer is that the geometry upon inspection is non-Euclidean; it is, as we saw before, intrinsically curved, and so angles and distances aren't preserved in the same way as they are in flat surfaces. This effect would happen in the opposite way if instead of living on the Earth we lived on some sort of massive Pringle potato chip, however that might work. All of this is represented in Figure 16.

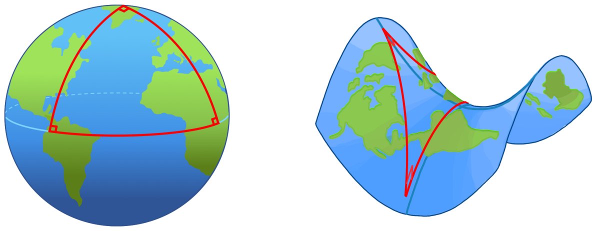


Fig. 16 - Diagram to show the angles in a triangular path within positively and negatively curved surfaces.

Stunningly, this property is what allows us to study things like the curvature of the universe from the inside of it [7].

The Gaussian curvature idea was also extended by his student, Riemann, a work which Einstein used in his study of space-time curvature [1].

Only the crust left to eat

From margherita to manifolds, differential geometry has proven itself fruitful in explaining the most intuitive every-day tricks just as well as the most subtle topological intricacies. It is no wonder a fascinating, far-reaching subject that - just like pizza - doesn't lack its own very personal charm and charisma.

References

[1] Rovelli C. Seven Brief Lessons on Physics. [Sette brevi lezioni di fisica] Trans Carnell S, Segre E. UK: Penguin Books; 2014.

[2] Pressley A. Elementary Differential Geometry. 2nd ed. London: Springer; 2010.

[3] Abate M, Tovena F. Curves and Surfaces. [Curve e superfici] Trans Gewurz DA. 1st ed.: Springer; 2006.

[4] Finat J., Workshop on Differential Geometry and its applications in Engineering Fields. 05-02-2019.; 2019.

[5] Venkateshan K. "Esoteric" Art of Shape Analysis — Level 2 — More on Curvatures. Medium. 2019. [Accessed 29-12-2019].

[6] Ballmann W. The Geometry of Submanifolds. Introduction to Geometry and Topology. 1st ed. Springer; 2018. pp. 143-144.

[7] Bhatia A. How a 19th Century Math Genius Taught Us the Best Way to Hold a Pizza Slice. WIRED. 2014. [Accessed 09-12-2019].

[8] Callens S, Zadpoor A. From flat sheets to curved geometries: Origami and kirigami approaches. Materials Today. 2017.

All figures were created by the author using Power Point and Notability.