String loop corrections in bosonic string path integrals

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Why do String Theory?

Interaction potentials in gauge theories (such as the quark confinement potential in the Standard Model) are governed by a class of quantum operators called Wilson loops.

These are incredibly difficult to compute. However, the gauge-string duality (an example of which is the famous AdS-CFT conjecture) says that these can be represented by partition functions in string theory.

Our particular research has focused on calculations which are directly relevant to tests of this conjecture. Furthermore, the bulk of the work has been calculating functional determinants which are endemic to high energy physics.

String Partition Functions

The partition function corresponding to a Wilson loop C in string theory is defined by the path integral [1]:

$$< W(C) >= Z[C] = \int DXDh_{ij}\exp\{(-S(h,X))\}.$$

Here, we integrate over all surfaces embedded in 26-dimensional Euclidean space carrying a metric h.

S(h,X) is the Polyakov action, defined by [2]:

$$S(h,X) = \int_\Sigma d^2\xi\sqrt{h}h^{ab}\delta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu.$$

Many configurations (h,X) we integrate over correspond to an equivalent action S(h,X) due to symmetry. We must get rid of this degeneracy by defining the measure properly!

Defining the Measure

We redefine the integration over the metrics h by breaking it up:

$$\delta h_{mn} = \underbrace{(2\delta\sigma + \nabla^p\delta v_p)\, h_{mn}}_{\text{Weyl rescaling } = \delta\sigma'} + \underbrace{(P_1\delta v)_{mn}}_{\text{Diffeomorphisms}} + \underbrace{\delta t^i T_{i,mn}}_{\text{Teichmüller}}$$

where Teichmuller deformations are all metric deformations which are not diffeomorphic nor conformal.

This allows us to rewrite the measure as:

$$\int Dh_{ij} \to J\int \frac{D\sigma' Dv dt^i}{N}.$$

Where J is a Jacobian corresponding to the breaking up of the measure, and N is a normalisation constant.

The Path Integral

Having redefined the measure, we compute the Jacobian factor J and the path integral reduces to:

$$\int_{\text{Teichmüller}} \frac{d\text{WP}}{\text{Volume}[\text{Ker}(\hat{P}_1)]\,||\text{Mapping Class Group}||}\sqrt{\det{}'\hat{P}_1^\dagger\hat{P}_1}\left(\frac{2\pi}{\int_M d^2\xi\sqrt{g}}\det{}'\Delta_g\right)^{-13}\Bigg|_{\hat{g}}.$$

Here everything is evaluated after having reduced the metric h to a conformally flat metric g. The integral is now reduced to a finite-dimensional integral over the Teichmuller space of the surface. The relevant factors are:

$d\text{WP}$ This is the Weil-Petersson measure (the measure on the Teichmuller space of the surface).

$\sqrt{\det{}'\hat{P}_1^\dagger\hat{P}_1}$ This is the determinant of an elliptic operator, which is equivalent to the Laplacian operator if g is conformally flat.

$\det{}'\Delta_g$ This is the determinant of the Laplacian operator on the surface, the most defining feature of the path integral.

References:
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Determinants of differential operators

How does one take the determinant of a differential operator on a Riemann surface? Recall that the determinant of a (1,1)-tensor is defined as the product of its eigenvalues. The eigenfunctions of a differential operator L on a Riemann surface form an infinite-dimensional Hilbert space i.e., there are a (countably) infinite number of eigenvalues. The infinite product will look like

$$\prod_{i=1}^{\infty}\lambda_i\,,$$

and it will not be well defined. To deal with this we define the zeta function

$$\zeta(s) = \sum_1^\infty \lambda_i^{-s},$$

which is well defined for large s. We then formally define the determinant of the operator of interest as

$$\det \mathrm{L} = e^{\zeta'(0)},$$

where the zeta function is defined at zero by analytic continuation. This definition can be motivated within pure mathematics e.g in Ray and Singer [3], and was first used in theoretical physics by Hawking [4].

These determinants are either extremely difficult or impossible to compute for two main reasons:

- The eigenfunctions of an operator are impossible to compute unless the Riemann surface is highly symmetric (i.e. a rectangle or sphere)
- The analytic continuation of arbitrary zeta functions is very difficult.

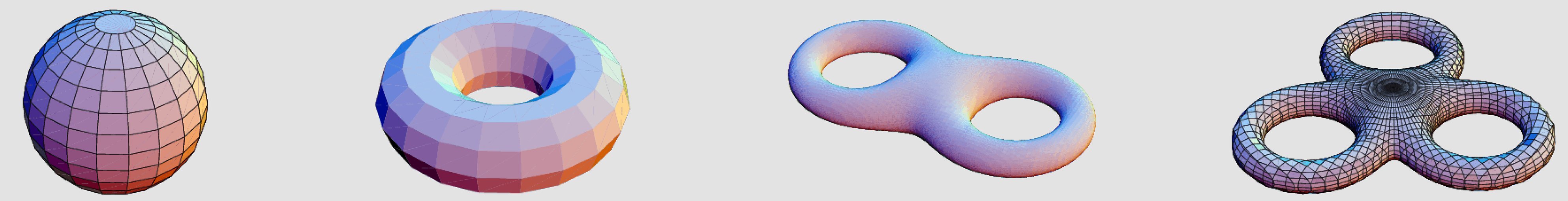

Fig 1. The compact Riemann surfaces can be topologically classified into the sphere, torus, double-torus, triple-torus etc [5].

The one-loop closed string path integral

We evaluated the partition function for the case where the world sheet has no boundary and the topology of a torus. We did this in a novel manner, and obtained the same result as first obtained by Polchinksi [6]:

$$\int \frac{d^2\tau}{4\pi\tau_2^2}\frac{|\eta(\tau)|^{-48}}{(2\pi\tau_2)^{12}}.$$

Here, τ is a complex number which parametrizes the moduli space of the torus. It is related to Teichmuller deformations of metric which cannot be induced by conformal rescalings or diffeomorphisms connected to the identity. The integral is taken over a fundamental region. Lastly, η is the Dedekind eta function, defined as

$$|\eta(\tau)| = \left|e^{i\pi\tau/12}\prod_{n>1}(1-e^{2i\pi n\tau})\right|.$$

Annulus correction to circular Wilson loop partition function.

We performed in a novel manner a calculation of the partition function where the world sheet is bound by a circular Wilson loop and has the topology of an annulus. We assume Dirichlet boundary conditions on the circular Wilson loop, and Neumann on the 'dynamical' inner boundary of the annulus. The determinant of the Laplacian on the annulus was computed in a new manner, using a contour integral method to sum eigenvalues [7] and agrees with calculations obtained by a different method [8]. Our final result is

$$W = \int_0^\infty d\tau|\eta(2i\tau)|^2\left(\frac{\tau}{4\pi}\exp\left(\frac{\pi\tau}{6}\right)\tilde{\beta}(e^{-4\pi\tau})\right)^{13}.$$

Now the moduli space is parametrized by a single real number τ. We have again used the Dedekind eta function, whilst the function denoted by $\tilde{\beta}$ is defined as

$$\tilde{\beta} = \prod_{k=1}^{\infty}(1+x^m)^{-1}.$$