Quantum Entanglement: We Are Inseparable!

CID:

Word Count: 2935

1. Challenge Accepted!

1.1 Classic Father-in-Law Problem

"Will you marry me?"

"Yes!"

That was how Alice and Bob should have started living happily ever after. What bothers the couple is that Alice's father Charles is not the biggest fan of this marriage.

"So, you are a what, a physicist?"

"You earn peanuts."

That was the response from Charles. He has a point though. Alice is the daughter of the man who owns the biggest company in the entire country. She deserves someone better than with her current fiancé in Charles' perspective.

"I don't give a damn about what he says. He doesn't even care about me at all! We haven't talked for three years! And now what, he just wants to split us apart?"

Bob has not seen Alice this angry since the time he did not reply to her message because he fell asleep in his laboratory. "But we do need his approval to survive in this country. Honey, he is the most powerful person in the country. If he is not happy about us, we can't even find a job here…" Bob knows too well that they are in no position to fight against his father-in-law.

"But I love you! That's what matters, right? We are like the same person now! I'd rather go with you to another country than being separated from you! We are inseparable!"

"That must be it, babe!" Out of nowhere Bob shouted excitedly! "We might be able to trick him to think that we are indeed inseparable!" Bob quickly kissed Alice on her lips and sprinted out of the room.

Alice is left confused, but she has faith in Bob, and knows that he will come up with something to protect their relationship.

1.2 The Brilliant Idea

Ever since Bob rushed out of home, he already spent three days in his laboratory. During this time, he came up with a plan to demonstrate that Alice is only 'telepathic[1]' with him, so they are meant to be together. Of course, he finished the first step, which is sending an arrogant message to Charles, together with proposed challenges and provoke Charles to give them the chance for this demonstration. Charles is no fool though, he consulted the best statisticians in the entire country before laughing at Bob confidently:

"Sure. I will no longer be your problem if you pass the challenges. But if you fail any one of them, I will make sure you never see my daughter again! See you next Sunday!"

So far so good. Bob sent the challenges to Alice and got back to work in his lab. He has a lot of hardware to prepare and theories to explain to Alice.

2. Unexpected Difficulty

2.1 The Challenges

Alice nervously opens Bob's message. She is quite relieved that Bob has a plan. However, she cannot possibly be happy with the challenges Bob proposed:

To show that Alice and Bob are the only correct person to one another, these challenges will demonstrate their telepathic ability to communicate to each other (with supporting equipment) even in different isolated locations. Charles is allowed to set-up the isolated locations and ask for repeated wins from Alice and Bob to ensure they did not get lucky.

Challenge 1:

Alice and Bob freely say the number $|0\rangle$ *or* $|1\rangle$ *repeatedly to each generate a series. The two series must be EXACTLY same.*

Challenge 2:[2]

Alice's father is allowed to give Alice and Bob each a number, either $|0\rangle$ *or* $|1\rangle$*. Again, Alice and Bob each say the number* $|0\rangle$ *or* $|0\rangle$ *in response.*
The rules are:

- *(i) If they are given* $|1\rangle|1\rangle$ *and reply* $|1\rangle|1\rangle$*, they WIN for once.*
- *(ii) if they are given* $|0\rangle|1\rangle$*, they cannot reply* $|0\rangle|1\rangle$*, or they FAIL immediately.*
- *(iii) if they are given* $|1\rangle|0\rangle$*, they cannot reply* $|1\rangle|0\rangle$*, or they FAIL immediately.*
- *(iv) if they are given* $|0\rangle|0\rangle$*, they cannot reply* $|1\rangle|1\rangle$*, or they FAIL immediately.*

2.2 Alice's Classical Considerations

Alice quickly considered both challenges and tries to figure out what Bob is trying to achieve here. For the first challenge, since the four equally probable conditions for each pair of digits are $|0\rangle|0\rangle$, $|0\rangle|1\rangle$, $|1\rangle|0\rangle$ and $|1\rangle|1\rangle$, and only $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$ satisfy the rule, they have a chance of 50% to obey the law for each pair of digits. Smart as Alice, she comes up with an obvious solution than randomly shouting out numbers: she can ask Bob to keep saying $|0\rangle$ and she can do the same to satisfy the condition. Cool! That is quite an easy solution. She quickly sends a message to Bob illustrating her idea.

But she got concerned quickly after. "This is in no way demonstrating telepathy, though. Maybe Bob has some other ideas."

What about the second challenge then?

This is way more difficult to achieve. In this scenario, both Bob and her need to adjust their answers according to the unknown digit Charles send to the other person. Alice tries to come up with a solution in her mind:

"The only way we win this thing is we both say $|1\rangle$ when we are both given $|1\rangle$, so the most straight forward answer is to say $|1\rangle$ when seeing a $|1\rangle$."

It looks like she is on the right track:

"I'm going to put myself as first digit since my name comes first alphabetically. If I see a $|1\rangle$ and say $|1\rangle$, and Bob sees $|0\rangle$, he must say $|1\rangle$ to avoid violating rule(ii). So, in this case he must always say $|1\rangle$."

Things are looking dangerous to her now:

"But hang on, if he always says $|1\rangle$, I cannot say $|0\rangle$ when I see a $|0\rangle$, to avoid breaking rule(iii). However, I cannot say $|1\rangle$ either because we need to avoid breaking rule(iv).

Something went wrong here."

She tries to adjust the starting point here:

"Relax, we still can find a way. No one said we have to say $|1\rangle|1\rangle$ every time we are given $|1\rangle|1\rangle$. Say if I say $|0\rangle$ when I see $|1\rangle$..."

But she immediately found the problem:

"Oh good, this way we will never win. It is getting really messy here. Let me right this down then." She picks up a pencil and started drawing a table on the wall, since Bob used all the refills to play Tic-Tac-Toe with his chemist friend last night.

Alice easily finished the table and is impressed with her wonderful handwritings:

What AB Sees \ What AB Says	00	01	10	11
00	OK	OK	OK	OK
01	OK	LOSE	OK	OK
10	OK	OK	LOSE	OK
11	LOSE	OK	OK	WIN!

But apart from that, she really has nothing to be happy about. She kept trying to figure out a way to win sometimes without losing, but with no luck. "Maybe ask Bob", she thought, "Oh right! Has he replied to my strategy about Challenge 1 yet? It's half an hour ago. If he hasn't replied, like the time he fell asleep in lab, he is so going to sleep on the coach tonight."

There is indeed an unread message in her phone, but not from Bob:

"We are not stupid. He won't be able to communicate with you until after you finish the challenges on Sunday. Oh, by the way, you need to win both challenges 100 times without failing to pass the challenge. Charles."

3. No Time to Give Up

3.1 Delivery from Bob

Alice cried a little bit before she fell asleep. She hates the fact that her father has the power to imprison Bob. Now what, how can she accomplish these two challenges on Sunday?

Not much progress has been made until Saturday night. All Alice can do is list all the possible outcomes out for both challenges. She gave up passing the challenge with certainty long times ago, and aim to figure out the strategy to pass the challenge with highest probability. For challenge 1 she is going to say $|0\rangle$ all the time, which gives 50% probability to pass the test if Bob puts in the same thinking. This is obviously the best strategy, since randomly shouting out numbers and have 100 pairs of digits being the same has a desperate probability of $\frac{1}{2^{100}}$, roughly 10^{-30}. For challenge 2, the best strategy is keep saying $|1\rangle$ and hope they are never given $|1\rangle|1\rangle$.

"If both of us always saying $|1\rangle|1\rangle$, to win 100 times we need approximately 300 trials without given $|0\rangle|0\rangle$, the probability is 0.75^{300}, which is about…"

"Deliveries from Mr Bob to Ms Alice?" Alice's train of thought is interrupted by a man shouting outside the door. She opens the door and sees a big cardboard box full of little cubes. "Here you go, Ms Alice." The delivery man puts the box down in the living room. Alice takes a careful look about the little cubes, they are labelled '1-1', '1-2' to '1-1000' and '2-1', '2-2' to '2-10000'. Another cube is labelled without number but with the word 'PETE[3]'. Alice stares at the word 'PETE' and it suddenly rings a bell. She knows this is a Hadamard gate[4,5]!

3.2 The Challenge Day

Alice did not sleep at all last night. She spent the entire night figuring out what Bob's plan is, and she is confident that she understands his plan. She got picked up to an abandoned factory, where she will be doing the challenges without seeing anything outside the empty factory. For preparation, Alice moved the cupboard full of little cubes in the room. Her dad is by her side, supervising her.

"Off you go, my daughter." He seems relaxed.

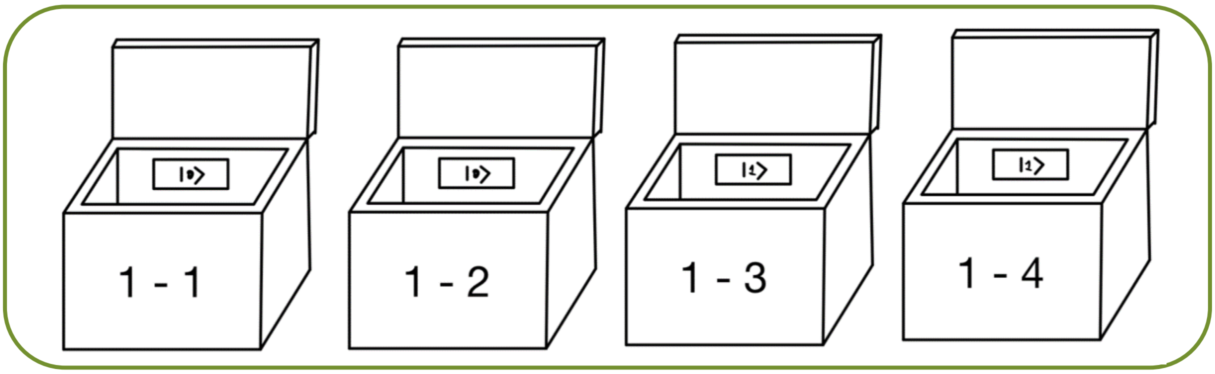
A few days ago, he saw Bob writing a lot of $|0\rangle$s on pieces of paper and put them in some boxes when preparing challenge 1. He planned to disavowal with their first demonstration if Alice just got lucky and keeps saying $|0\rangle$s.

The first challenge is easy to operate on Alice's side. She simply opens the cube labelled '1-1' and reads out the number written on a paper in the cube.

"First digit is $|0\rangle$." She speaks. A faint sneer of satisfaction crossed her father's face.

Then she opens the cube '1-2', '1-3', '1-4' and keeps shouting out the number in the cube.

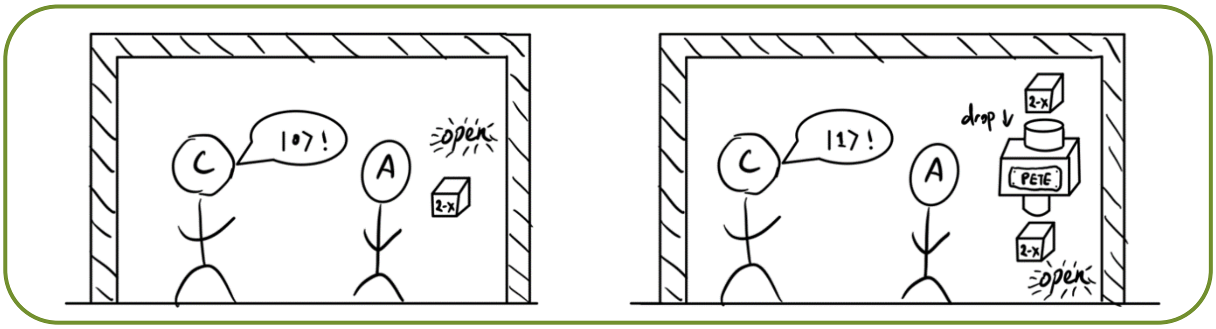
"And then$|0\rangle$, and then $|1\rangle$, and then $|1\rangle$…"



Charles is surprised. Alice did not just keep saying $|0\rangle$s for 100 digits. Instead, she is really saying a series that seems random. While doubting if the couples are really telepathic, he records all the numbers Alice said and keeps the series to be compared with Bob's series.

The second demonstration takes more time from Alice since she is holding the PETE box now. Charles is not quite sure what the PETE box does, but he starts to see a pattern. Whenever he says a $|0\rangle$, Alice just opens the cube labelled '2-x'; and whenever he says a

‘1’, Alice would throw the cube ‘2-x’ in the PETE box, wait for the cube to come outside and open the cube. She then reads out whatever is on the note in the cube. Her behaviour seems quite odd for a psychic, but he is fully convinced that Alice has no way knowing what Bob sees and what he says, since he has his best scientists to set up the isolated rooms.



These two challenges are very time consuming to conduct. By the time Alice finished her part it is already the end of the day. The supervisor of Bob reported Bob’s outcomes for both challenges, together with his strange behaviours when he during the challenges. Charles cannot help but noticed Bob did exactly what Alice did for both challenges. Maybe what they do does make them telepathic!

Just as Alice and Bob planned, after Charles compared Alice’s answer with Bob’s answer, it is no doubt that they passed both challenges! This is quite unbelievable to him, but he is still a man of honour, so he acknowledged their telepathic ability and left the couple live happily ever after.

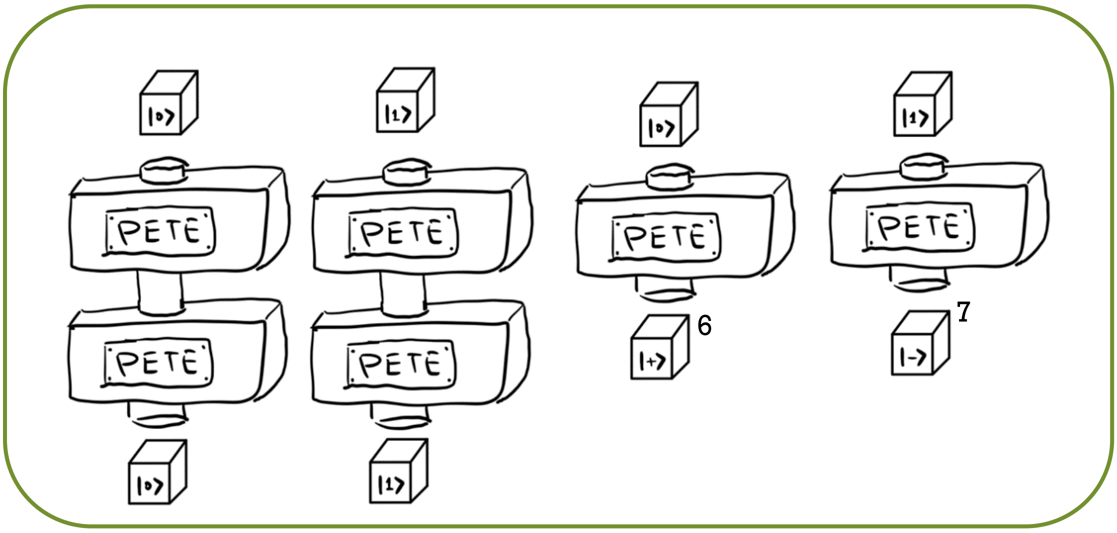
4. 30 years later

4.1 The Magical PETE box

Charles is lying on bed at home. Surrounding him are the greatest doctors in the entire country. He knows that there is not much time left for him to live, but he still has one thing in mind. He wants to know how Alice and Bob achieved telepathy.

Bob passed away a few years ago, so Alice is the only one with an explanation. She does not hate her dad anymore, and since Bob is no longer here, she feels free to tell her father all about the challenges.

“Let’s start from the PETE box, shall we?” She knows that it is the part Charles has no clue about. “It has this magical ability, such that when a number pass through two PETE boxes, nothing happens to the number; but if the number only passes through one PETE box, then the number becomes a ‘0’ or ‘1’ with equal probability.”



This is already confusing to Charles. He cannot possibly believe something like that exists in the world.

"I know you are confused, dad. But it does exist! It is called a Hadamard gate in quantum computing, and is practically made already." Says Alice.

"The way to understand this is to think about this in mathematical form. It transforms a $|0\rangle$ or a $|1\rangle$ to superpositions of $|0\rangle$ and $|1\rangle$, namely $|+\rangle$[6] and $|-\rangle$[7] respectively. With this property, we can make quantum entangled states. And when we measure a superposed state (by opening the cube and read the note inside), the state collides down to only one of the compositions of the state, so either $|0\rangle$ or $|1\rangle$."

Charles seems like he noticed the trick here: "So if a $|0\rangle$ goes through the PETE box twice, it becomes…" He starts drawing on a piece of paper[8]:

$$|0\rangle \rightarrow |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)] = |0\rangle$$

$$|1\rangle \rightarrow |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \rightarrow \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)] = |1\rangle$$

"That's how it works! And it works the same with $|1\rangle$! How about the negative sign then? How come I never see a note in the cube with a $-|1\rangle$?"

"We cannot measure negative signs, dad. It is just not something physical. I will just keep going on…"

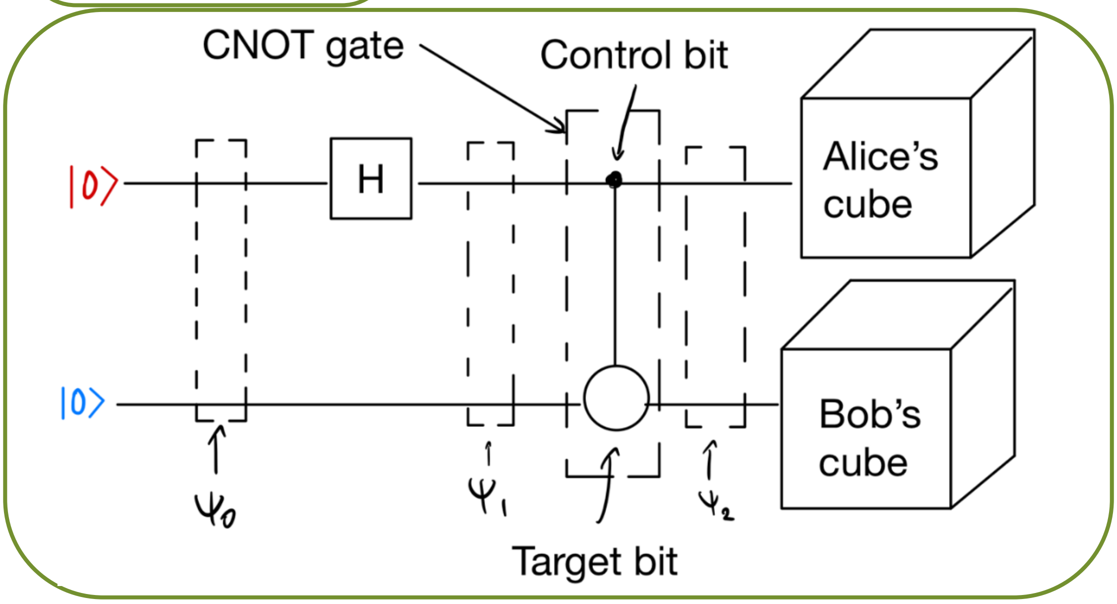
4.2 The Entangled State

"What Bob did in the lab was to use PETE boxes, and another operator call the CNOT gate[9,10]. The CNOT gate acts like…"

Input	Output
00	**0**0
01	**0**1
10	**1**1
11	**1**0

"So basically, the first quantum bit (qubit) (written in bold in the above table) is called 'control bit', which is not changed through the gate. The second one is a 'target bit', which changes between '0' and '1' if the control bit is '1', and remains the same if the control bit is '0'."

She started drawing as well. "Then Bob did something like this…"



It is easy to start with:

$$|\psi_0\rangle = |0\rangle|0\rangle$$

After the Hadamard gate (PETE box) (labelled 'H'), the two qubits become:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)$$

The two qubits then go through the CNOT gate, with the first qubit being the 'control bit'. Therefore, the first qubit remains the same. The second qubit acts depending on the first bit, so:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

which is the quantum entangled state we are looking for.

"So in this case, he achieved something with state $|0\rangle|0\rangle + |1\rangle|1\rangle$ and threw this state into the two cubes (with red state in Alice's cube and blue in Bob's) we opened during the challenge. If either one of us measures our state, by opening the cube, the other state is then determined. This type of state is called an entangled state."

"So, this is how you achieved the first challenge. I see, I see. What about the second one then? Let me think about it…" Charles thinks he knows the trick now.

"If I design something like this…" He comes up with a solution:

$$|\psi_3\rangle = |0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle$$

"Good job, dad! That is indeed what he did. Remember how we throw the cube through the PETE box if we hear $|1\rangle$ and open the box directly if we hear $|0\rangle$? If you calculate carefully…"

"If we hear $|0\rangle|0\rangle$, we will both open the cube directly, measuring one of the states from $|\psi_3\rangle$. If only I hear $|1\rangle$, then…"

$$|\psi_4\rangle = \frac{1}{2^{1.5}}[(|0\rangle + |1\rangle)|0\rangle + (|0\rangle + |1\rangle)|1\rangle + (|0\rangle - |1\rangle)|0\rangle] = \frac{1}{2^{1.5}}[2|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|1\rangle]$$

"If only Bob hears $|1\rangle$…"

$$|\psi_5\rangle = \frac{1}{2^{1.5}}[|0\rangle(|0\rangle + |1\rangle) + |0\rangle(|0\rangle - |1\rangle) + |1\rangle(|0\rangle + |1\rangle)] = \frac{1}{2^{1.5}}[2|0\rangle|0\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle]$$

"And if we both hear $|1\rangle$…"

$$|\psi_6\rangle = \frac{1}{2^2}[(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) + (|0\rangle + |1\rangle)(|0\rangle - |1\rangle) + (|0\rangle - |1\rangle)(|0\rangle + |1\rangle)]$$
$$= \frac{1}{2^2}(3|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle)$$

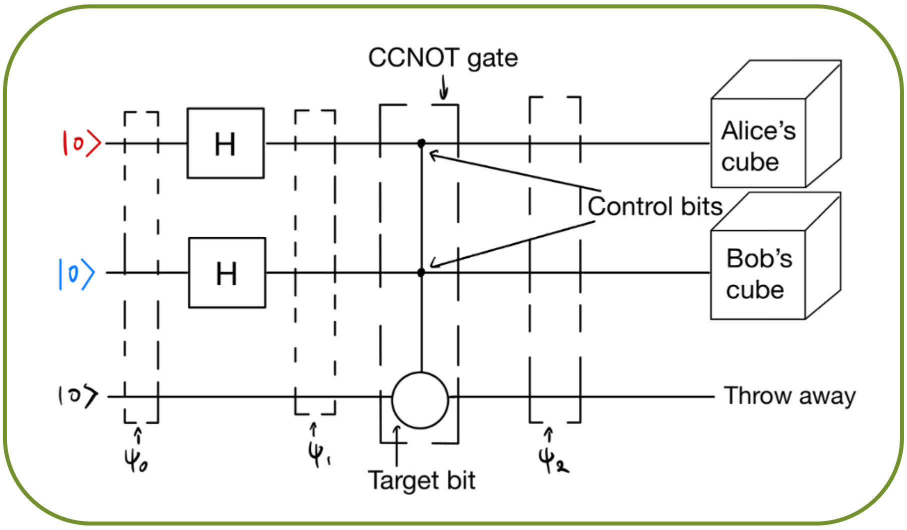
Inputs	Outputs
000	000
001	001
010	010
011	011
100	100
101	101
110	111
111	110

"All the rules are followed at all time!" Alice explains to Charles with excitement.

"How did he achieve this entangled state though?" Asked Charles.

"He used this thing called a CCNOT gate[11,12], which has two control bits, and the target bit only changes when both control bits are '1', just like shown in this table…"

"And then designed this circuit…"



Just like before we start with:

$$|\psi_7\rangle = |0\rangle|0\rangle|0\rangle$$

$$|\psi_8\rangle = \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)|0\rangle = \frac{1}{2}(|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|0\rangle)$$

$$|\psi_9\rangle = \frac{1}{2}(|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle)$$

"It is easy to see that first two qubits of $|1\rangle|1\rangle|1\rangle$ is not what we want in $|\psi_3\rangle$, so we simply measure the third qubit. If we see a '0', then we put the first two balls in Alice and Bob's cubes; if we see a '1', we toss all three qubits away."

It took Charles a while to go through the whole thing, but he eventually came to a clear understanding of it. "I'm so sorry you had to figure these all out by yourself. That must have been the longest night for you. I'm so proud of you for being so brilliantly smart!" Charles, for the first and last time, gave Alice his compliment.

"Like father, like daughter." Alice smiles with tears in her eyes.

The End

References:

[1] It was mentioned in my plan feed back to check the idea of 'telepathy', especially the non-locality of it. Thanks to Dr Steve Kolthammer's recommendation of this paper:

Shalm, L., Meyer-Scott, E., Christensen, B., Bierhorst, P., Wayne, M., Stevens, M., Gerrits, T., Glancy, S., Hamel, D., Allman, M., Coakley, K., Dyer, S., Hodge, C., Lita, A., Verma, V., Lambrocco, C., Tortorici, E., Migdall, A., Zhang, Y., Kumor, D., Farr, W., Marsili, F., Shaw, M., Stern, J., Abellán, C., Amaya, W., Pruneri, V., Jennewein, T., Mitchell, M., Kwiat, P., Bienfang, J., Mirin, R., Knill, E. and Nam, S., 2015. Strong Loophole-Free Test of Local Realism. *Physical Review Letters*, [online] 115(25). Available at: <https://journals.aps.org/prl/pdf/10.1103/PhysRevLett.115.250402> [Accessed 3 January 2022].

[2] based on: Rudolph, T., n.d. *Q is for quantum*.

[3] Rudolph, T., n.d. *Q is for quantum*.

[4] **This book teaches the operation of Hadamard gate in details which are not covered in this article** Nielsen, M. and Chuang, I., 2021. *Quantum computation and quantum information*. Cambridge: Cambridge University Press.

[5] **This shows the existence of Hadamard gates** Tipsmark, A., Dong, R., Laghaout, A., Marek, P., Ježek, M. and Andersen, U., 2011. Experimental demonstration of a Hadamard gate for coherent state qubits. *Physical Review A*, 84(5).

[6] $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, **here refers to superposition of** $|0\rangle\ and\ |1\rangle$, **the meaning of plus sign will become clearer later.** Nielsen, M. and Chuang, I., 2021. *Quantum computation and quantum information*. Cambridge: Cambridge University Press.

[7] $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, **here refers to superposition of** $|0\rangle\ and\ |1\rangle$, **the meaning of minus sign will become clearer later.** Nielsen, M. and Chuang, I., 2021. *Quantum computation and quantum information*. Cambridge: Cambridge University Press.

[8] **The constants for the below equations are for normalisation purposes only. Feel free to ignore them for now since they are not related to this article.**

[9] **Set-up of a CNOT gate:**

2022. [online] Available at: <https://www.researchgate.net/figure/Experimental-setup-for-the-demonstration-of-the-CNOT-gate-without-any-path-interference_fig3_7387485> [Accessed 3 January 2022].

[10] **How CNOT gates behave:** Nielsen, M. and Chuang, I., 2021. *Quantum computation and quantum information*. Cambridge: Cambridge University Press.

[11] **Existence of a CCNOT gate:**

Yu, N., Duan, R. and Ying, M., 2013. Five two-qubit gates are necessary for implementing the Toffoli gate. *Physical Review A*, 88(1).

[12] **How CCNOT gates behave:**

Nielsen, M. and Chuang, I., 2021. *Quantum computation and quantum information*. Cambridge: Cambridge University Press.