SCALAR FIELD ENTANGLEMENT ENTROPY IN A 2D CAUSAL SET

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Motivation for the Project

It is now largely believed that entanglement entropy provides a better understanding of quantum gravity. One of the approaches to the latter, Causal Set Theory (CST), has seen a recent growth in interest. While the entanglement entropy for a causal set was expected to match its continuum analogue, instead of the continuum area law, a spacetime-volume law was obtained. To resolve this discrepancy, an ultraviolet (UV) cutoff has to be imposed, which is naturally provided in CST contrarily to most other approaches.

Causal Set Theory

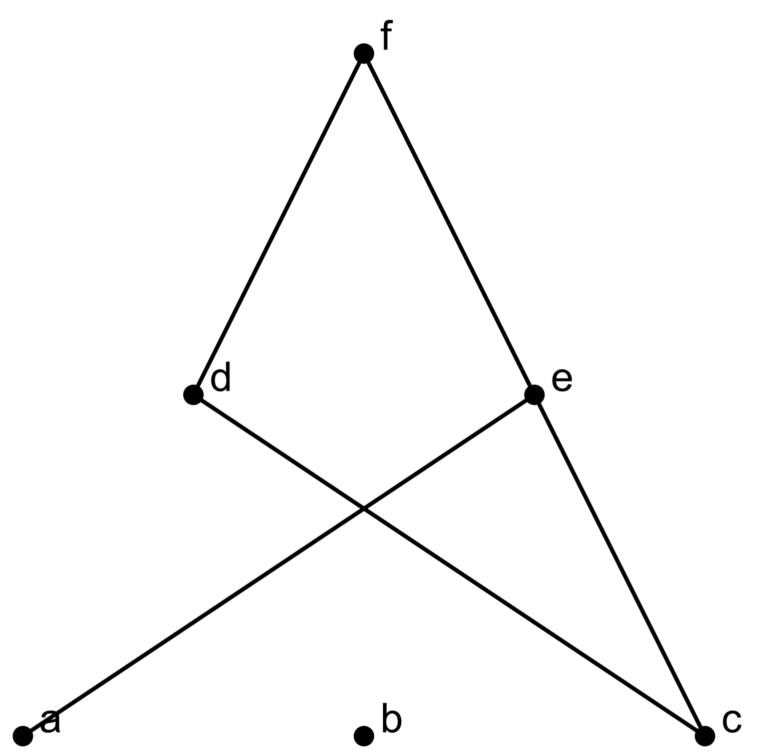
CST is founded upon two key concepts: causal order, and the discreteness of spacetime. A locally finite partially ordered set, $\mathcal{C}$, is formed by the discrete elements and the ordering relation defined by the causal structure. The causal set (or causet) is the pair $(\mathcal{C}, \preceq)$, where $\preceq$ is the partial order relation, and satisfies:

- Reflexivity: for all $u \in \mathcal{C}$, $u \preceq u$.
- Antisymmetry: for all $u, v \in \mathcal{C}$, $u \preceq v \preceq u$ implies $u = v$.
- Transitivity: for all $u, v, w \in \mathcal{C}$, $u \preceq v \preceq w$ implies $u \preceq w$.
- Local finiteness: for all $u, v \in \mathcal{C}$, $|[u, v]| < \infty$, where $[u, v] := \{w \in \mathcal{C} \,|\, u \preceq w \preceq v\}$ is a causal interval and $|\mathcal{A}|$ denotes the cardinality of a set $\mathcal{A}$.

A causal set can then be represented by the causal matrix, C, with matrix elements given by [1]:

$$C_{xy} = \begin{cases} 1 & \text{for } x \prec y \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

or diagrammatically by a Hasse diagram, where elements of $\mathcal{C}$ correspond to points and if two elements u and v are related as $u \preceq v$, then u is positioned below v with a line connecting them as shown below in Fig. 1.



$$\begin{pmatrix} 0&0&0&0&1&1 \\ 0&0&0&0&0&0 \\ 0&0&0&1&1&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \end{pmatrix}$$

Figure 1:An example of a Hasse diagram and the corresponding causal matrix.

Continuum Entanglement Entropy

The Von Neumann entropy is given by

$$S = \mathrm{Tr}\rho \ln \rho^{-1} \tag{2}$$

with ρ being the density matrix for a Cauchy hypersurface Σ. By dividing Σ into two complementary subregions A and B as shown in Fig. 2, one can obtain the reduced density matrix for A

$$\rho_A = \mathrm{Tr}_B \rho. \tag{3}$$

By then substituting (3) into (2), the entropy associated with A is:

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A = \sum_\lambda \lambda \ln|\lambda| \tag{4}$$

where λ are the eigenvalues of the reduced density matrix in the continuum. This would be the entanglement entropy between A and B had there been a pure initial ρ [1]. One would obtain the same result by starting with subregion B instead of A and calculating S_B [2].

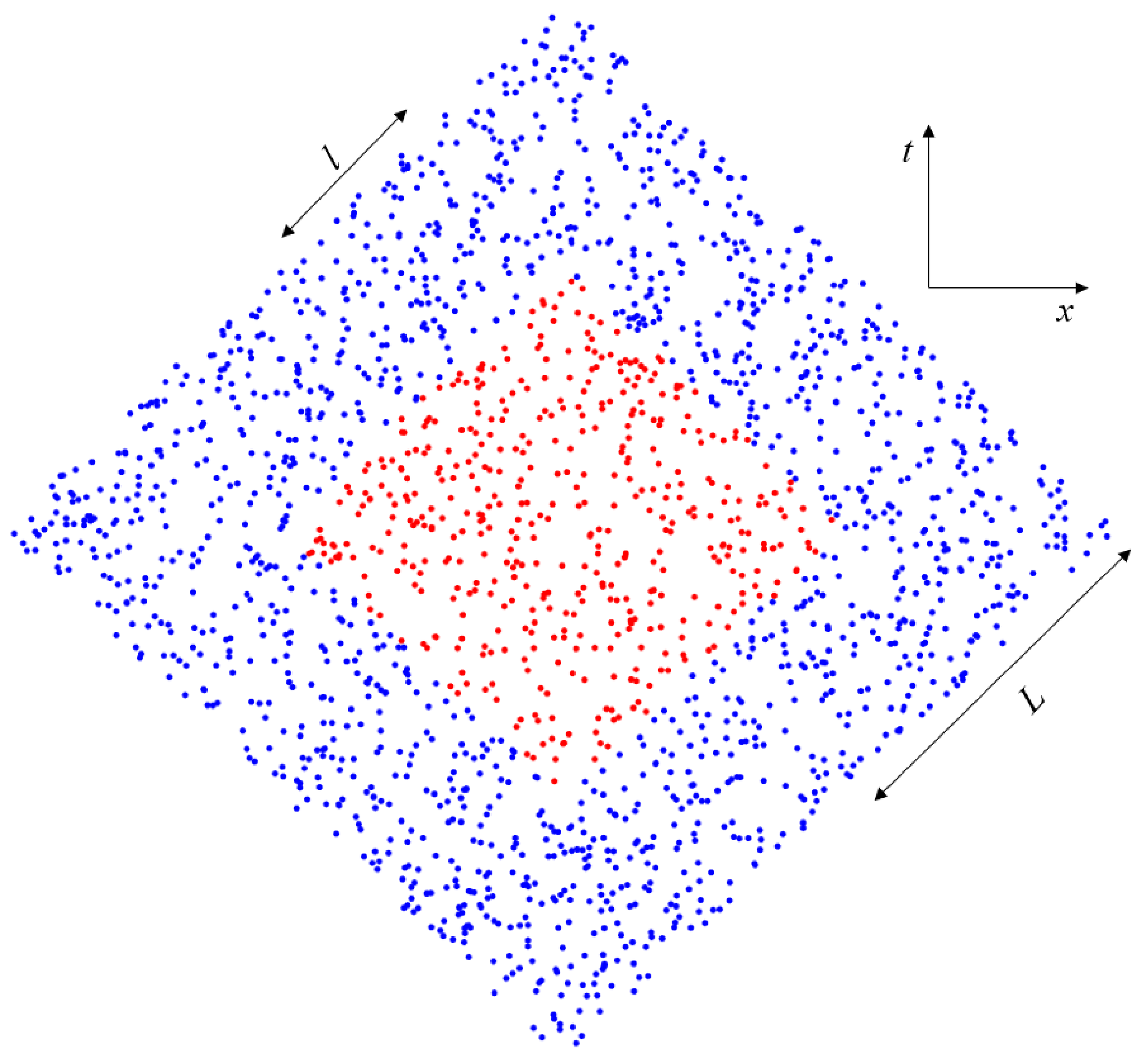


Figure 2:Causet diamond nested within larger diamond of 2000 elements generated via a Poisson sprinkling. L and l correspond to the half-lengths of the large diamond and the small diamond respectively.

From [3], the entropy in a massless scalar field and a massive scalar field is expected to scale with the UV cutoff, a, the subregion half-length, l, and the mass, m, as:

$$S \sim \frac{1}{3} \ln\left(\frac{l}{a}\right) \tag{5}$$

and

$$S \sim -\frac{1}{3} \ln\left(ma\right) \tag{6}$$

respectively.

CST Entanglement Entropy

To compute entanglement entropy in causets, an equivalent of (4) must be found. Using the Sorkin-Johnston prescription, the Wightman function W (corresponding to a pure state) is first found by taking the positive part of the Pauli-Jordan function, defined as in the continuum by:

$$\Delta(x, y) = G_R(x, y) - G_R(y, x) \tag{7}$$

and whose spectrum in the CST is shown in Fig. 3 where $G_R(x, y)$ is the retarded Green function. From [1], G_R for a scalar field in a 1+1d causal set is:

$$G_R(x, y) = \frac{1}{2} C \left(I + \frac{m^2}{2\rho} C \right)^{-1} \tag{8}$$

where C is the causal matrix given in (1) [1].

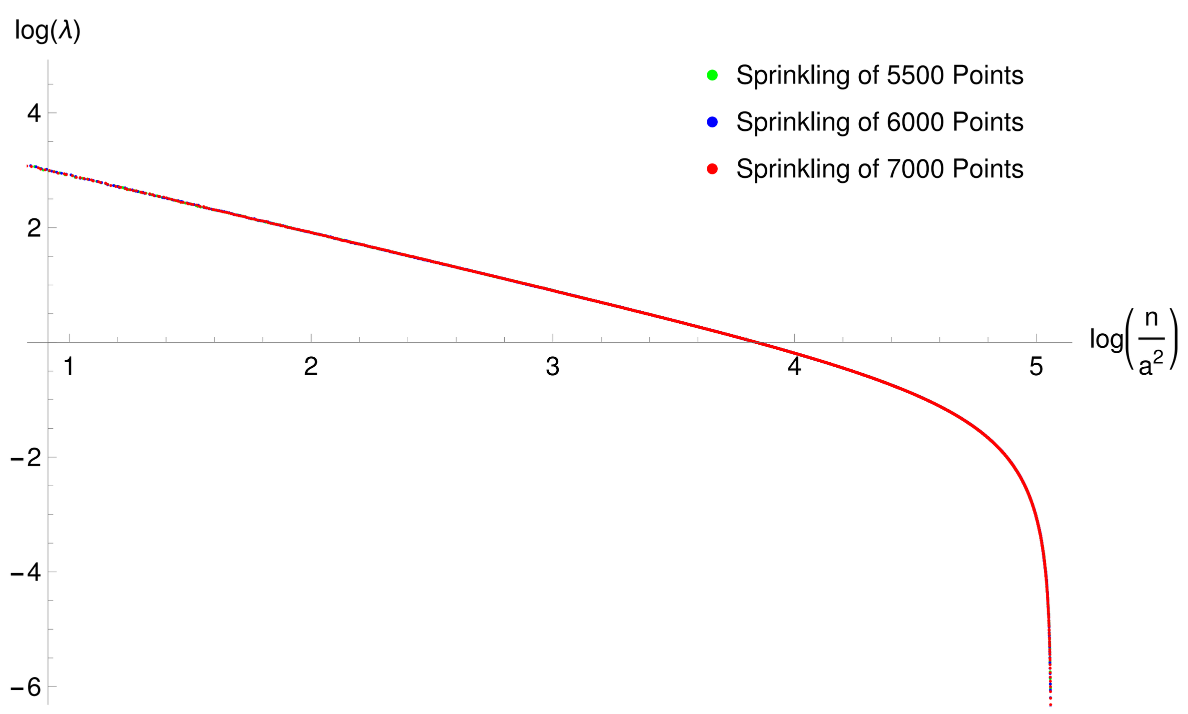


Figure 3:Collapse of the spectra of the Pauli-Jordan function (used to find the entropy) for different densities with a being the cutoff and n labelling the eigenvalues.

The entanglement entropy is then found by substituting λ in (4) by the eigenvalues of $\Delta^{-1} W$. If one applies this calculation without any restriction, the entanglement entropy obeys a spacetime-volume law – it grows linearly with N. This result does not match the area law continuum expectation of (5). Furthermore, calculated in this way, the causal set entanglement entropy is also found to be two orders of magnitude larger than the continuum equivalent [2]. These two issues suggest the need for a truncation.

Covariant Truncation Scheme

A vital part of getting physical results from (4) is to exclude functions in the kernel of $i\Delta$ for which the eigenvalues are undefined. Unfortunately, this operation is much harder in causets than in the continuum due to the presence of numerous small eigenvalues as seen in Fig. 3. In fact, if the smallest eigenvalue retained has magnitude $\widetilde{\lambda}_{min} \sim \sqrt{N}$ and a truncation is performed in both the larger and the smaller diamond, then the continuum logarithmic behaviour is obtained.

Results and Headway

The entropy scalings obtained are shown below in Fig. 4 for a massless field and Fig. 5 for a massive scalar field with the coefficients 0.335 ± 0.007 and -0.333 ± 0.012 being in close agreement with the continuum expectations of $\frac{1}{3}$ and $-\frac{1}{3}$ respectively.

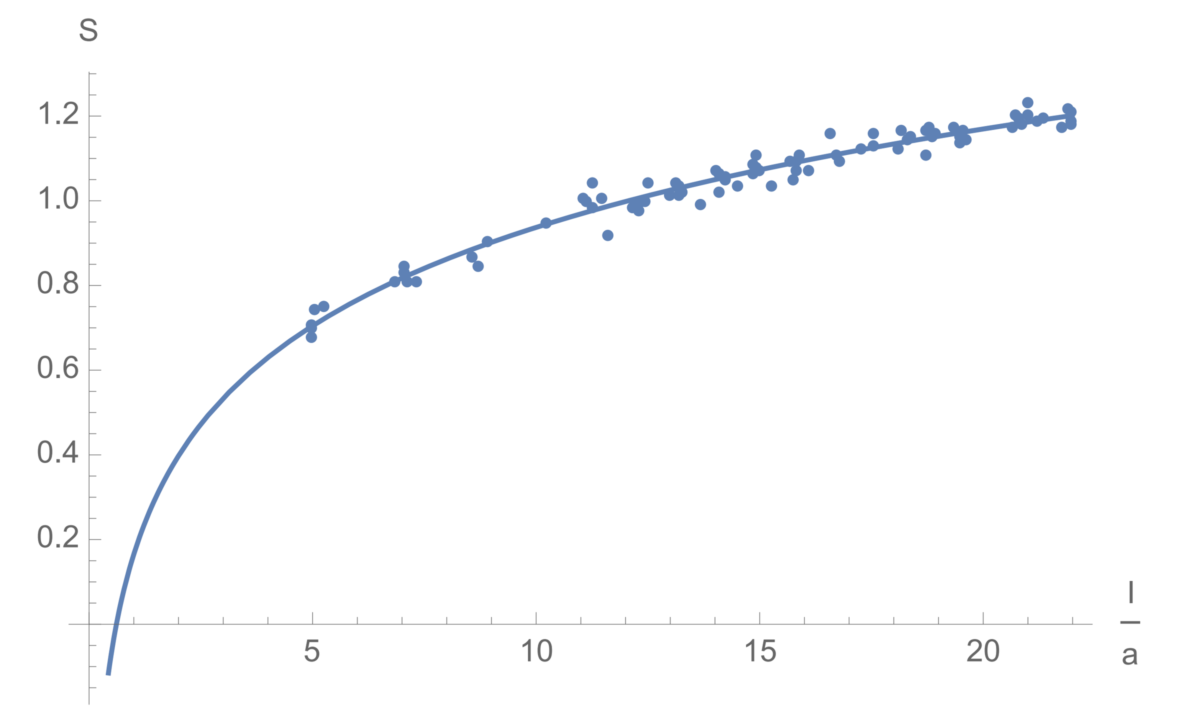


Figure 4:Fit of $S = 0.335 \log(l/a) + 0.165$ for sprinklings of up to 19000 points massless causal sets with a cutoff of λ_{min}

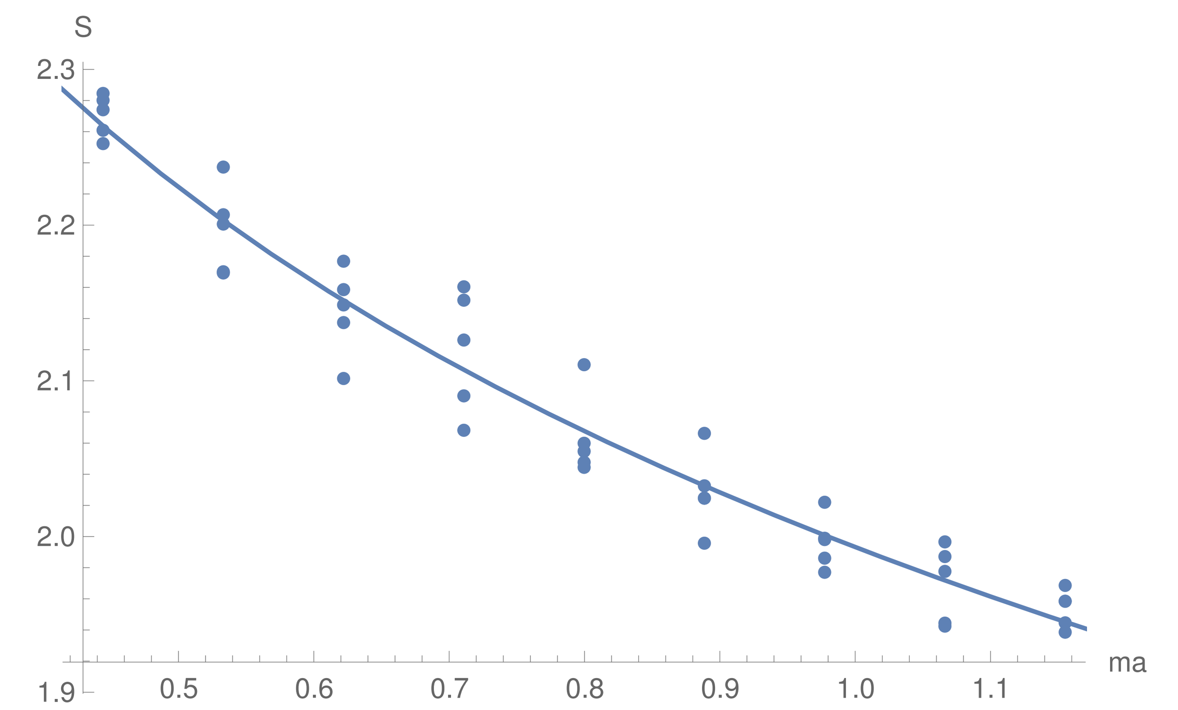


Figure 5:Fit of $S = -0.333 \log(ma) + 1.993$ for sprinklings of 20000 points massive causal sets with a cutoff of λ_{min} and masses ranging from 5 to 13.

One of the major findings of the project was that instead of the continuum entropy area law, a spacetime-volume law was obtained for causal sets. This discrepancy was related to the small but finite parts of the spectrum of $i\Delta$. The expected continuum area law was recovered after truncation. Keeping the above discoveries in mind, one can start to see the light and comprehend entanglement entropy in CST. This raises questions concerning the entropy related to black hole horizons and whether the latter is partially or even wholly entanglement entropy.

References

[1] S. P. Johnston, "Quantum fields on causal sets," Ph.D. dissertation, Imperial College London, Sept, 2010.

[2] Y. K. Yazdi, "Entanglement entropy of scalar fields in causal set theory," 2017.

[3] P. Calabrese and J. Cardy, "Entanglement entropy and quantum field theory," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2004, no. 06, Jun 2004.