Integer Quantum Hall Effect

Word Count: 2839

Introduction

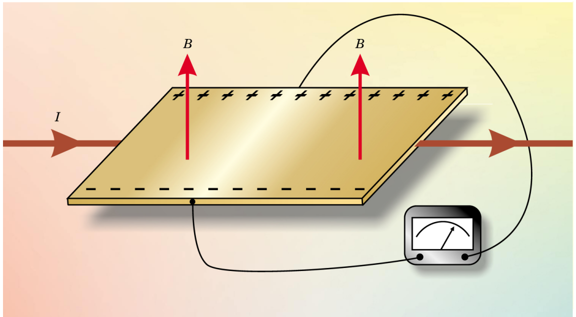


Figure 1: Hall used a gold leaf as the conductor in the experiment that found the Hall Effect. Image reproduced from [1].

Our story began in 1879, when a young PhD student by the name of Edwin Hall, performed an experiment at John Hopkins University [1]. As shown in Fig. 1, he drove a current through a thin gold leaf, while a magnetic field was being applied perpendicular to the metal plane. When he connected a voltmeter to the edges of the gold leaf, he found a transverse voltage perpendicular to both the current and the magnetic field. This is the effect that is being taught to almost every undergraduate physicist in the world today – the Hall Effect.

A century later, when Klaus von Klitzing repeated the experiment with some modifications, little did he expect that it became the source that inspired the development of much more fascinating physics. The modifications he made were mainly in two aspects [2]. First, he performed the experiment at a temperature near absolute zero, at only $T = 1.5K$. Also, the semiconductor used, Si MOSFET, constrained the electrons in it tightly along the z-direction, rendering it a 2D plane for the electrons to move in. When he applied the magnetic field in z-direction and drove a current through the semiconductor in x-direction, a Hall voltage appeared just as expected, but when he performed quantitative analysis, the result differed from the Hall effect in a surprising way.

Classical Expectations

To fully appreciate the result of the experiment, we need to reshape Ohm's law:

$$\vec{E} = \rho \vec{J}, \tag{1}$$

where $\vec{E}$ and $\vec{J}$ are quantities that the reader may be familiar with – electric field strength and current density. The resistivity, ρ, however, needs a bit of modification. Because the current in one direction, say x-direction, can result in a y-direction voltage, ρ here needs to be a matrix rather than a single number, called resistivity tensor. Because of rotational symmetry around z-axis, the resistivity tensor, ρ, is of the form:

$$\rho = \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{bmatrix} \text{[3]}, \tag{2}$$

where ρ_{xx} is the resistivity that correlates the voltage and current in the same direction, known as longitudinal resistivity and ρ_{xy} is the resistivity that correlates the voltage and current in different directions, called Hall resistivity.

In classical physics, the longitudinal resistivity is expected to be insensitive to the magnetic field strength and maintain a constant value, whereas the Hall resistivity is expected to have a simple linear relationship with magnetic field [3]. This is not surprising, as one would expect the bigger the magnetic field, the bigger the Hall resistivity.

Integer Quantum Hall Effect (IQHE)

Von Klitzing measured the Hall and longitudinal resistivities, when he was varying the magnetic field and the result shows a trend as this:

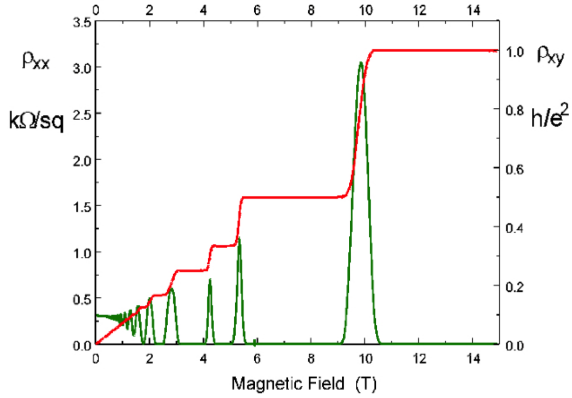


Figure 2: The Hall resistivity shown by the red line displays plateaus, while the longitudinal resistivity shown by the green line vanishes at the plateaus. Although this is not the original result of von Klitzing's experiment, it showed a similar trend. This plot was chosen because it displayed the trend more clearly. Image reproduced from [4].

Both the Hall and longitudinal resistivities seem to behave as expected at low magnetic field strength: the former increased linearly, while the latter remained almost constant. Then, the Hall resistivity started to display plateaus, during which the longitudinal resistivities become zero. Before moving on to a higher plateau, there happened to be a jump in both the Hall and longitudinal resistivities.

Von Klitzing recognized that these peculiar plateaus do not appear at random values, but rather share a common pattern:

$$\rho_{xy} = \frac{h}{e^2}\frac{1}{\nu} \quad \nu = 1, 2, 3... \, [3, \ 5] \tag{3}$$

and the Hall conductivity has an exact quantization on these plateaus:

$$\sigma_{xy} = \frac{1}{\rho_{xy}} = \frac{\nu e^2}{h} \quad \nu = 1, 2, 3... \, [3, \ 5], \tag{4}$$

where h is the Plank constant and e is the elementary charge. Because ν takes integer values, this is referred to as the Integer Quantum Hall Effect (IQHE) [3]. Von Klitzing recognized this quantization and wrote on his experimental record. It was said that it was this page of record that won him the Nobel Prize in Physics in 1985 [6]. The value of resistance defined as

$$R_K = \frac{h}{e^2} = 25812.80745... \, \Omega \, [7] \tag{5}$$

is known as the von Klitzing constant.

Microscopic Explanation

Free Electron Model

To understand the reason behind the plateaus, the free electron model developed by Arnold Sommerfeld in 1927, can be employed here [8]. As the name suggests, the electrons in this model are treated as if there were free electrons without any interactions with the ions. The ions are only there to maintain net electric neutrality throughout the material. The interactions between electrons are also mostly ignored, except the requirement imposed by the Pauli Exclusion Principle [9]: no two electrons can occupy the same state. However, the principle does not mean that no two electrons can have the same energy: they can be in two different states with the same amount of energy.

Landau Levels

Classically, an electron in a magnetic field will perform orbiting motion in a circle, called cyclotron orbit. The energy of the electron can be arbitrary, but the angular frequency of the motion, known as the cyclotron frequency, ω, takes the definite value of

$$\omega = \frac{eB}{m} \, [3, \; 10], \tag{6}$$

where B is the magnetic field strength and m is the mass of an electron.

It would not be entirely surprising to learn that the physics behind IQHE involves quantum mechanics, so some elements of quantum mechanics have to be added to the fold [3]. One important property that electrons possess is particle-wave duality, which indicates that not only do we need to consider the electrons as particles like the analysis made in the classical way, but the electrons' orbits also need to accommodate them being waves as well. Therefore, electrons' orbit circumferences have to be integer multiples of their wavelengths, such as the left-hand side of Fig. 3, whereas orbits like the one on the right-hand side are prohibited. The above semi-classical analysis gives us an idea of electrons' quantized behaviors, but with a full quantum mechanical calculation, which was made in the 1930s by the Soviet physicist Lev Landau, the energy of the orbits was calculated to be

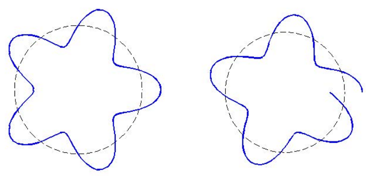


Figure 3: Orbit circumferences have to be integer multiples of wavelength to accommodate wave behaviors of electrons. Image reproduced from [11].

$$E_n = (n + \frac{1}{2})\hbar\omega \, [3, \; 9], \tag{7}$$

where n is an arbitrary natural number labeling energy levels from lowest to high and $\hbar$ is the reduced Plank's constant. These energy levels are called Landau Levels.

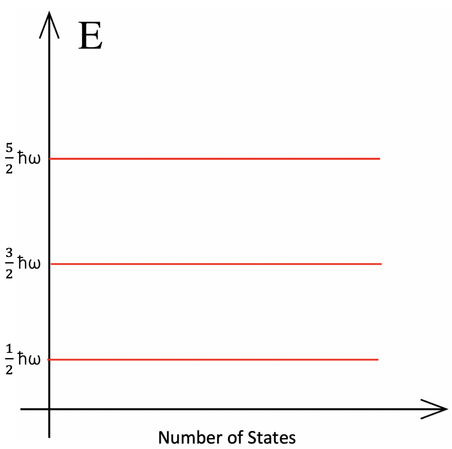


Figure 4: The number of states is dependent on the magnetic field strength, but independent of energy. Image reproduced from [3].

The number of states that can be accommodated by each energy level turned out to be directly proportional to the magnetic field strength and equal among all the energy levels as shown in Fig. 4. With the help of the Pauli Exclusion Principle, a microscopic picture can be established by imagining the lines of different energies as a strangely shaped bottle. When one starts to fill electrons to the states, it is like pouring some water into the bottle. The water first fills into the bottom of the bottle, to populate the lowest energy states. When the first line is completely filled, the water will have no other choice but to populate the second lowest energy level. The process continues until all the water or electrons are filled into a state.

Edge States

The above analysis applies to electrons that are well within the edges of the sample, but there is an interesting effect for electrons that locate near the edges [3]. For the electrons to be confined within the sample, there has to be a steep rise in potential to prevent the electrons from escaping. In a semi-classical picture, the effect of this potential on the electrons can be seen as walls that when hit by electrons, will bounce the electrons back into the sample. Because the cyclotron orbits are bound to maintain a clockwise or counterclockwise orbital direction depending on the direction of the magnetic field, these electrons will have no other choice but to form skipping motions and moves along the edges like the red and blue lines in Fig. 5.

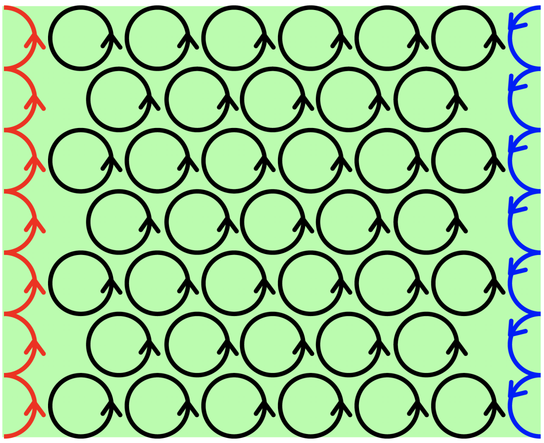


Figure 5: Electrons within the sample perform cyclotron motions, while the edges states form skipping orbits. Image reproduced from [12].

These edge states are special because they are very difficult to disrupt. The main source of energy loss in electricity transmission is electrons being bounced off their trajectories in some random directions. These edge states, on the other hand, have only one possible direction to move in, so the current they transport will yield very little energy dissipation [3].

As shown in Fig. 5, with the comprehensive microscopic picture of the electrons under the influence of the magnetic field established: the electrons far from the edges form quantized circular orbits, while the edge states form skipping motion, it is time to turn on the electric field and see the current.

Movement of the Centers

One important result that holds true in both classical and quantum physics is that when an external field is present in addition to the magnetic field, the center of the cyclotron orbits drifts along the equipotentials of the external field [3].

The states with the same energy can be organized in a fashion that they are evenly spaced along the y-axis for the electrons to fill [3, 10]. When an electric field is turned on in the y-direction, the trajectories of the centers follow the equipotentials of the electric field: along the x-direction, so a current in the x-direction is produced. Each fully filled Landau level will contribute current of $\frac{e^2}{h}V_{Hall}$

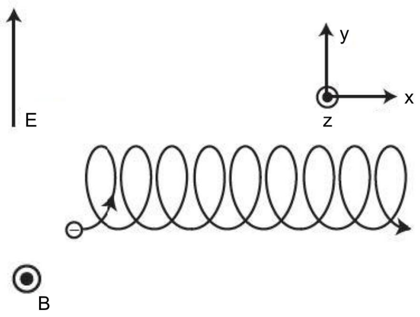


Figure 6: Electrons in cyclotron orbits drift along equipotentials under the influence of electric field. Image reproduced from [13].

[3, 10], which totals to $\frac{\nu e^2}{h} V_{Hall}$ and give the correct Hall conductivity and resistivity in equation 3 and 4.When ν Landau levels are filled, every available state that is less than the maximum energy at various y-coordinates is occupied, so a current along the electric field direction is not possible, which explains the zero value in ρ_{xx}. So the values of the resistivities on the plateaus perfectly correspond to ν filled Landau levels.

At this point, one may be tempted to say “mission accomplished!”, as we found a microscopic explanation of the plateaus. However, if one thinks about it carefully, it does not actually explain the plateaus, because only specific values of magnetic field strength correspond with the situations where ν Landau levels are completely full. In all other cases, the top Landau level will only be partially filled, resulting in a continuous change of Hall resistivity with magnetic field strength. The point of the plateaus is the Hall resistivity stays the same over some range of different magnetic field strengths. What went missing here?

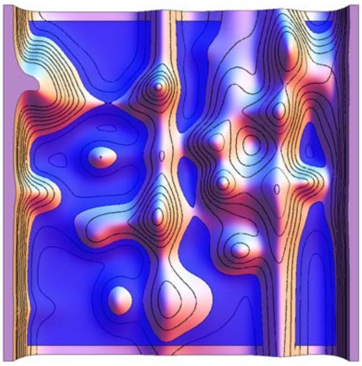
Role of Disorder

The one missing piece of our puzzle is disorder. Disorder is a terminology used to refer to the inevitable impurities in the sample being experimented on [3, 10]. It turns out that the straight, precisely quantized plateaus on the result owe their existence to the messy, usually undesirable disorder in experiments.

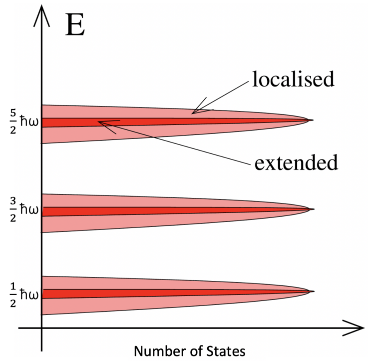
Disorder can be modeled as potentials added on top of the energy of the Landau levels [3, 10]. As a result of the disorder, these Landau levels lose their straight lines and turn into Gaussian shapes like in Fig. 7b. On the plot, one can imagine that the states with energies more deviated from the Landau levels would mean they are more affected by disorder, whereas states with energies around the Landau levels would mean they are less affected by disorder.

Disorders are mountains and pits on a smooth zero potential surface, causing the equipotential lines to wind around them like the one shown in Fig. 7a. As the centers of the circular orbits move along equipotentials, some electrons will be trapped to move around these mountains and pits and become localized. Of course, these localized states are the states more affected by the disorder. In contrast, the states with orbital centers that still generally managed to move from one side to the other, by following equipotentials that cross the valleys between mountains and pits are less affected by disorder and called extended states. The localized states are bound to follow the desperate fate of endlessly circulating around a mountain or a pit and cannot contribute to electricity conduction [10].

With the new ingredient of disorder added, let us see what it does to the resistance curve. Once again, we pour all the water into the energy levels, but when the magnetic field strength is decreased, the capacity of each Landau level decreases as well. As a result, the water will be forced to take up higher Landau levels. Rather than starting to populate the next Landau level, it will first occupy the



(a) The diagram illustrates an example of equipotentials of a sample with disorder. Image reproduced from [12].



(b) Landau Levels are widened by disorder. Image reproduced from [3].

Figure 7: The effect of disorder on the sample and its energy levels

localized states in between the two Landau levels, which does not contribute to electricity conduction and have no change to the current. However, once the extended states near the higher Landau level started to get filled, there will be electricity conducted through the conductor, which corresponds to a jump in both the Hall and longitudinal resistance. When we enter the next localized states directly above these extended states, by the same argument as before, there will be no electricity conduction, so the resistivities will be remaining on a plateau. While varying magnetic field strength, this cycle of jump, plateau, jump, plateau... will repeat, which explains the resistivity curve perfectly.

Beyond IQHE

Metrology

IQHE proved to have a wide-ranging impact in condensed matter physics and other fields as well. One of the most prominent examples is in the field of metrology.

After its discovery, various measurements of von Klitzing constant in different samples yielded an uncertainty of less than one part in 10^{10} [14]. The value in the high precision of quantized resistance was immediately recognized, especially in the field of metrology. The von Klitzing constant has a very direct connection to the fine structure constant, α, an important constant that measures the strength of electromagnetic interactions in Quantum Electrodynamics (QED):

$$\alpha = \frac{c\mu_0}{2R_K} \text{ [10]}, \tag{8}$$

where c is the speed of light and μ_0 is the vacuum permeability. At the time of the discovery of IQHE, von Klitzing recognized that a measurement of the constant is an independent measurement of the fine structure constant at very high precision. The original paper on which he published the discovery of IQHE was even titled *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance* [2] to emphasize the application of IQHE in this area.

Also, von Klitzing constant established a direct link between the two fundamental constants, the Plank constant, h, and the elementary charge, e, and the unit Ohm. Since 1990, the resistance of von Klitzing constant is used as the standard calibration for resistance throughout the world [15]. The high precision "triggered a realization of a new international system of units based on fundamental constants" [16] and culminated in the redefinition of SI units in 2019, completely based on fundamental constants [7]. With this redefinition, h and e became defined rather than measured, which resulted in von Klitzing constant having the exact value of 25812.80745... Ω [7].

More Quantum Hall Effects

The unexpected discovery of IQHE reignited physicists' interest in studying the Hall Effect in the quantum setting. Many variants of the Quantum Hall Effect was discovered since then.

Merely two years after the discovery of IQHE, Horst Störmer, and Daniel Tsui discovered a different type of Quantum Hall Effect, where the plateaus were even possible to appear at fractional values of ν in equation 3. They and Robert Laughlin, who provided a partial explanation for the phenomenon, was awarded the Nobel Prize in Physics in 1998. This new effect is called Fraction Quantum Hall Effect (FQHE) [10] and required a reexamination of the explanation provided for IQHE and led to fruitful discoveries, such as new quasiparticles [17] and anyons [18, 19], which are particles that are neither Fermions nor Bosons.

The exciting discoveries made during these 3 years 'marked a turning point in condensed-matter physics' [16]. The new ideas and discoveries it inspired go far beyond what this article can cover, but Fig. 8 attempts to illustrate a few of these. Two of the latest related phenomena observed in laboratories are Quantum Anomalous Hall Effect (QAHE) [20] and Quantum Spin Hall Effect (QSHE) [21], where the Quantum Hall plateaus were observed even without the need of an external magnetic field.

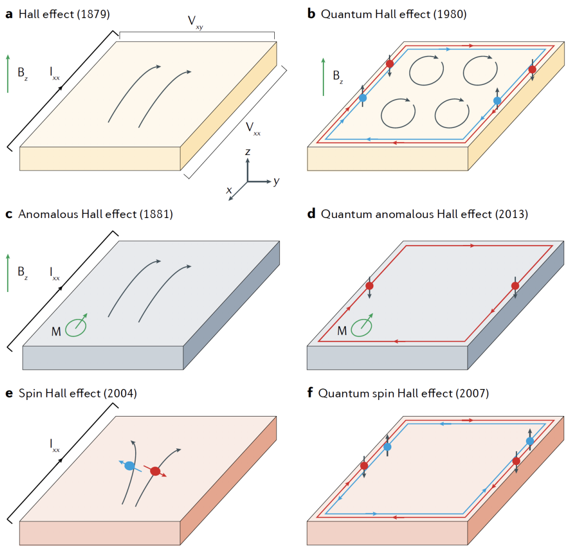


Figure 8: The diagrams show a few variants of Hall Effect and the year they were experimentally observed in laboratories. Among them, d, e and f are phenomena inspired by IQHE. Image reproduced from [16].

Topological Quantum Matter

Disorder plays an irreplaceable role in the realization of IQHE, but the specific form of disorder has little impact on the quantization value. Also, as the edge states only move in one direction, their motion demonstrates resilience toward disorder as well. All this robustness against the randomness of disorder in QHE indicates a topological protection. In fact, Laughlin provided a topological explanation of IQHE and showed that the quantization values of IQHE plateaus were resilient to not only disorder, but also different geometries of samples [22]. These topological properties also existed in the subsequent QAHE and QSHE. They form a new type of state of matter known as topological insulators. This angle of viewing Quantum Hall Effects through the lens of topology has now evolved into a new branch of condensed matter physics, known as topological quantum matter [23]. One of the most prominent achievements in this field was recognized by the award of Nobel Prize in Physics for the study of topological phase of matter in 2016 [24].

Concluding Remarks

The IQHE is, by its own rights, an interesting phenomenon. It is an example of a relatively familiar phenomenon displaying new properties in the quantum realm. However, in my opinion, the reason why this effect managed to attract the attention of scientists around the world is because of its wide-ranging and far-reaching influences beyond the effect itself. As this year is the 40 year anniversary of the discovery of IQHE, looking back at its history, physicists realize that it was only the beginning of an exciting and still on-going chapter of condensed matter physics [16].

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