Homogenized boundary conditions and resonance effects in Faraday cages

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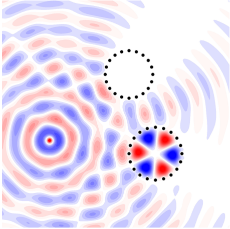

Joint work with: Jon Chapman, Nick Trefethen, Ian Hewitt (Oxford)

ACCA-UK/JP Workshop, Imperial College London
14th March 2017

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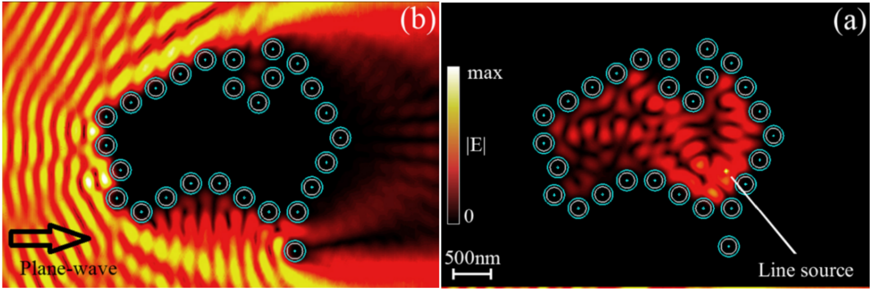
Scientists and engineers use the Faraday cage effect *all the time*, for electrostatic and electromagnetic **shielding**.

An everyday example: the wire mesh in the door of your microwave oven

A more exotic example: optical Faraday cages built from nanowires:



Optical Metacages, Mirzaei et. al, PRL 115, 2015

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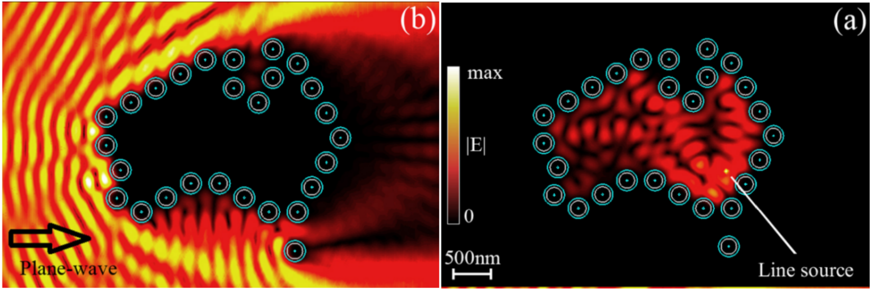

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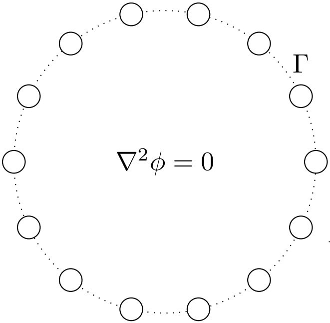
But it's surprisingly hard to find any quantitative analysis of the phenomenon!

2D electrostatic model

$\nabla^2 \phi = 0$



no net charge on cage
$\phi \sim \log(|z|) + o(1)$
as $z \to \infty$

external point charge
$\phi = \log(|z - z_s|) + O(1)$
as $z \to z_s$

z_s

M disks of radius r at equal (but unknown) potential ϕ_0

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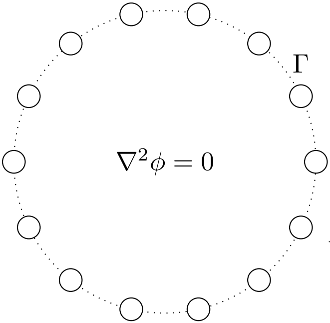

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Question: How does $|\nabla \phi|$ inside the cage behave as a function of M and r?

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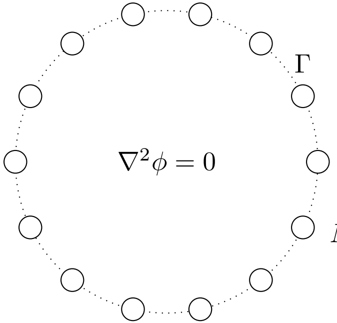

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Question: How does $|\nabla \phi|$ inside the cage behave as a function of M and r?

Answer: (Chapman, Hewett and Trefethen, SIAM Review 2015)

$$|\nabla \phi| = \mathcal{O}\left(\frac{|\log r|}{M}\right) \text{ for } M \to \infty \text{ with } r \ll 1/M.$$

Numerical solution

"Mikhlin method": approximate ϕ by a series expansion around the wire centres

$$\phi(z) = \log|z - z_s| + \sum_{j=1}^{M} \left(a_j \log|z - z_j| + \mathrm{Re}\left[\sum_{k=1}^{n_s} \frac{b_{j,k}}{(z - z_j)^k} \right] \right),$$

subject to (for the behaviour at infinity)

$$\sum_{j=1}^{M} a_j = 0.$$

The coefficients $a_j \in \mathbb{R}$ and $b_{j,k} \in \mathbb{C}$ are determined by imposing the boundary condition $\phi = \phi_0$ (recall that ϕ_0 is also an unknown) at a set of $n_c \geq M(2n_s + 1)$ collocation points on the wire boundaries.

Homogenized boundary condition

In the limit $M \to \infty$ with $r \ll 1/M$ we used the method of multiple scales to derive a homogenized boundary condition on Γ, leading to a continuum model:

$\nabla^2 \phi = 0$

$\phi \sim \log(|z|) + o(1)$
as $z \to \infty$

Γ

$\phi = \log(|z - z_s|) + O(1)$
as $z \to z_s$

$\nabla^2 \phi = 0$

z_s

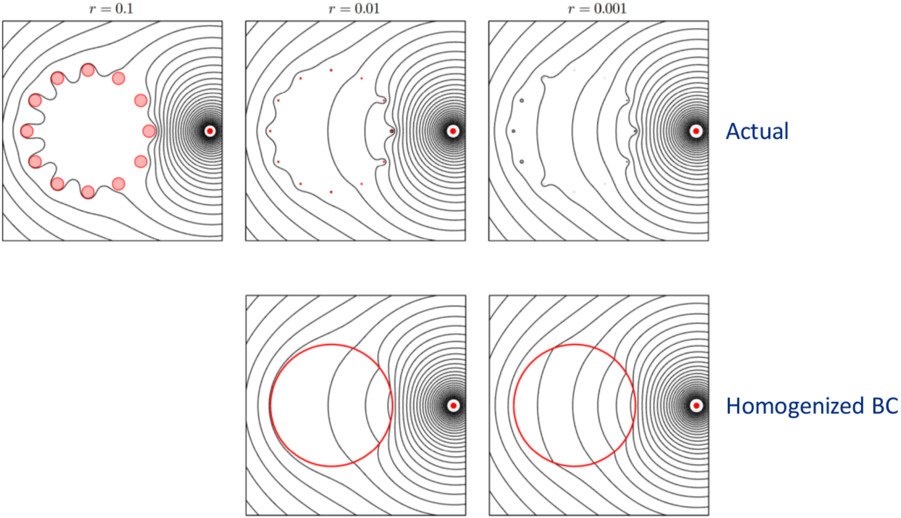
$$[\phi]_-^+ = 0$$

$$\left[\frac{\partial \phi}{\partial n}\right]_-^+ = \alpha(\phi - \phi_0)$$

$$\alpha = \frac{M}{\log 1/(rM)} \qquad \left(\text{So } \alpha = \mathcal{O}\left(1\right) \text{ when } r = \frac{\mathrm{e}^{-M/\alpha}}{M}\right)$$

Physical interpretation: α represents "capacitance per unit length".

Numerics vs asymptotics: $(z_s = 2,\ M = 12)$



$\alpha = -65.8$ $\qquad$ $\alpha = 5.66$ $\qquad$ $\alpha = 2.71$

Unanswered questions

What about thicker wires? ($r = \mathcal{O}\left(1/M\right)$ rather than $r \ll 1/M$)

What about non-circular wires?

What about the time-harmonic wave problem?
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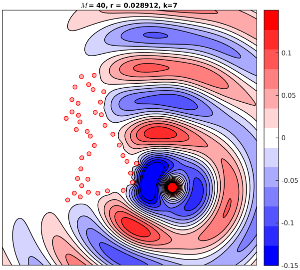
In the wave problem, "good shielding" means $|\phi|$ is small inside the cage

But one observes **resonance effects** where the presence of the cage can actually **amplify** the external field rather than shield against it!

Great British Faraday cage



nonresonant

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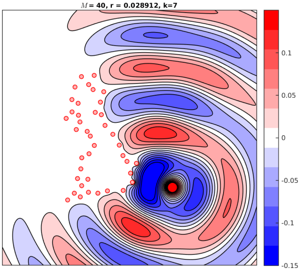

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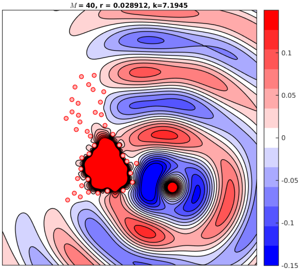

resonant

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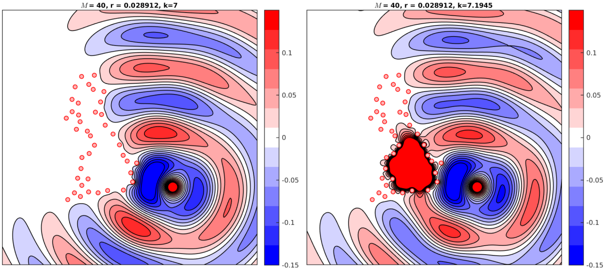
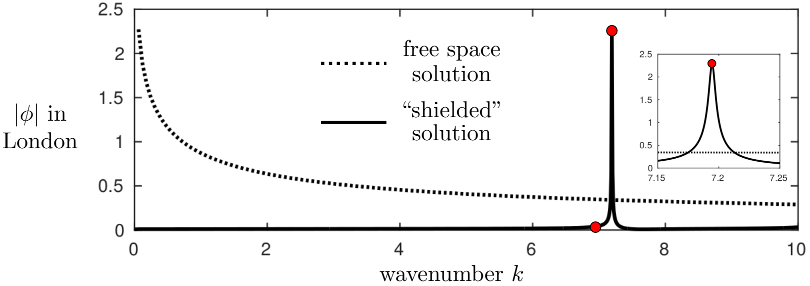
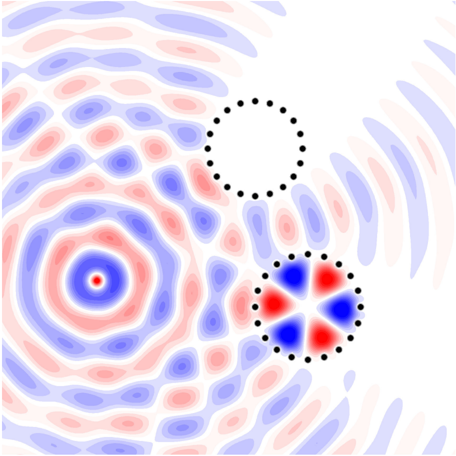


nonresonant resonant

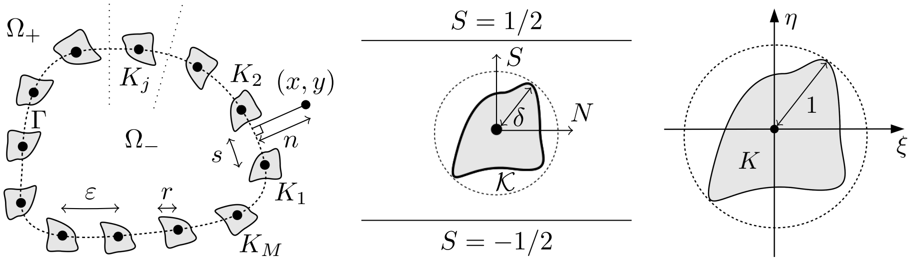


Another example

Two cages of slightly different radii irradiated by a point source, one shielding and the other amplifying:



Problem formulation



$$K_j = z_j + \mathrm{e}^{\mathrm{i}\theta_j}(rK), \qquad \varepsilon = |\Gamma|/M, \qquad r = \delta\varepsilon$$

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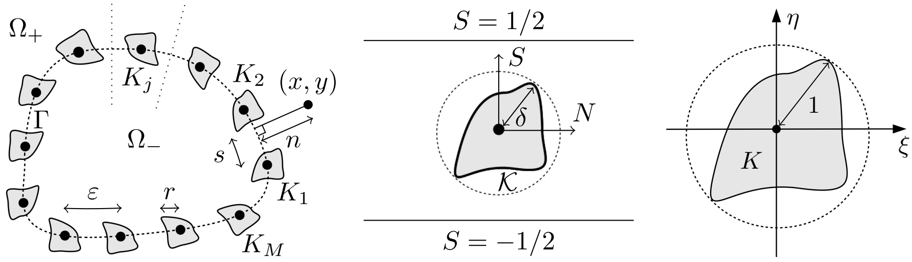

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We seek a complex-valued scalar field ϕ, outgoing at infinity, and satisfying

$$(\nabla^2 + k^2)\phi = f \qquad \text{in } D := \mathbb{R}^2 \setminus \bigcup_{j=1}^{M} K_j,$$

$$\phi = 0 \qquad \text{on } \partial K_j,\ j = 1, \ldots, M.$$

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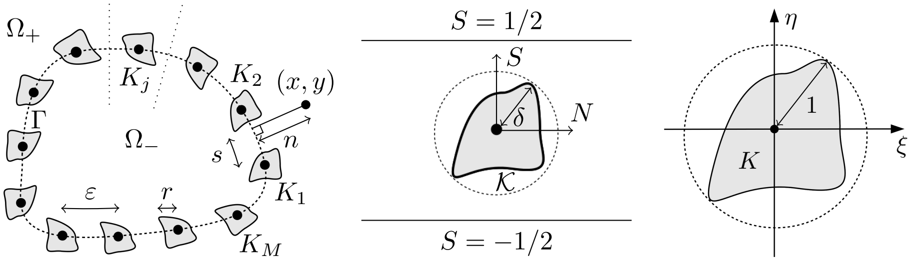

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Aim: investigate behaviour of ϕ as $\varepsilon \to 0$ ($M \to \infty$), for all $0 < \delta < \delta_* = \mathcal{O}(1)$. (Our previous analysis considered the case $k = 0$ and $\delta \ll 1$, K the unit circle.)

Homogenization for $\varepsilon \to 0$

We approximate the effect of the discrete wires by seeking an **outer solution** satisfying a **homogenized boundary condition** on the interface Γ, which we derive using the **method of multiple scales**. This is valid in the **low-frequency regime** $k = \mathcal{O}(1)$.

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- Sanchez-Hubert/Sanchez-Palencia (1982)
- Cioranescu/Murat (1997)
- Nazarov (2008 x 2)
- Delourme (2010), Delourme/Haddar/Joly (2012), Delourme/Haddar/Joly (2013), Claeys/Delourme (2013), Delourme/Schmidt/Semin (2015)
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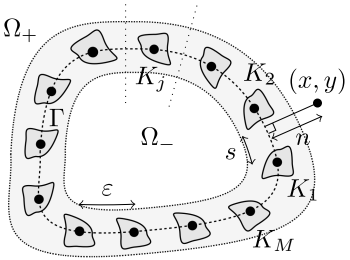
We note also the complementary work going on at Imperial College London (Craster, Maling, Schnitzer...) on "whispering Bloch waves" and other higher frequency phenomena.

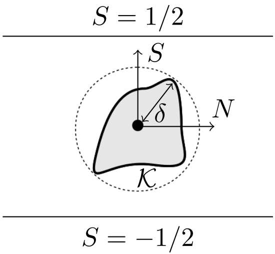
Homogenization for $\varepsilon \to 0$

We look for outer approximations in $\Omega_\pm$ of the form

$$\phi(x,y) = \phi_0^\pm(x,y) + \varepsilon\phi_1^\pm(x,y) + \mathcal{O}\left(\varepsilon^2\right) \qquad \text{in } \Omega_\pm$$

The rapid variation close to Γ is modelled by a boundary layer of width $\mathcal{O}(\varepsilon)$.





Here we look for a solution in multiple-scales form

$$\phi(n,s) = \Phi(N,S;s)$$

where (N,S) are the boundary layer variables defined by $(n,s) = (\varepsilon N, \varepsilon S)$, and $\Phi(N,S;s)$ is assumed to be 1-periodic in the fast tangential variable S.

Matching and cell problems

The outer limits ($N \to \pm\infty$) of the boundary layer solution Φ must match the inner limits ($n \to 0$) of the outer solutions $\phi^{\pm}$, which are

$$\phi_0^{\pm}(0, s) + \varepsilon \left(N \frac{\partial \phi_0^{\pm}}{\partial n}(0, s) + \phi_1^{\pm}(0, s) \right) + \mathcal{O}\left(\varepsilon^2\right)$$

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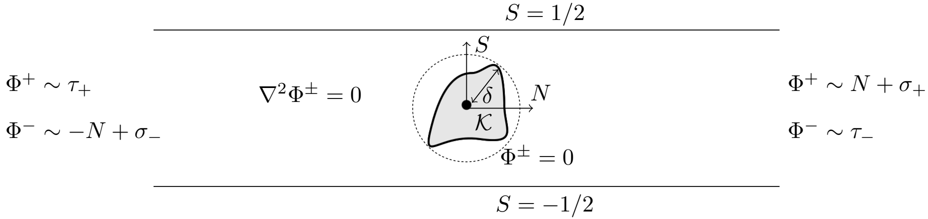
Hence

$$\Phi(N,S;s) = \varepsilon\Big(\frac{\partial \phi_0^{+}}{\partial n}(0,s)\Phi^{+}(N,S) - \frac{\partial \phi_0^{-}}{\partial n}(0,s)\Phi^{-}(N,S)\Big) + \mathcal{O}\left(\varepsilon^2\right),$$

where $\Phi^{\pm}$ satisfy the following cell problems:



with periodic boundary conditions on $S = \pm 1/2$

$$\boxed{\text{Given } K \text{ and } \delta\text{, we must determine the constants } \sigma_{\pm},\ \tau_{\pm}}$$

Matching conditions:

$$\varepsilon\sigma_+ \frac{\partial \phi_0^+}{\partial n} - \varepsilon\tau_- \frac{\partial \phi_0^-}{\partial n} = \phi_0^+ + \varepsilon\phi_1^+ \qquad \text{on } \Gamma,$$

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Thin wires ($\delta \ll 1$). In this case $\sigma_\pm, \tau_\pm \gg 1$, since

$$\Phi^\pm(N,S) \sim \frac{1}{2\pi}\Re\left\{\pm\pi Z + \log\left(2\sinh\pi Z\right) + \log\frac{1}{2\pi\delta} + a_0\right\}, \qquad Z = N + iS,$$
$$\sigma_\pm, \tau_\pm \sim \frac{1}{2\pi}\left(\log\frac{1}{2\pi\delta} + a_0\right) + \mathcal{O}\left(\delta\right), \qquad \delta \to 0.$$

Here a_0 is related to the **logarithmic capacity** of K, $c(K)$, by $a_0 = -\log c(K)$. For K the unit disc, $a_0 = 0$; for K a line segment of length 2, $a_0 = \log 2$.

In particular, there is a distinguished scaling in which $\sigma_\pm, \tau_\pm = \mathcal{O}\left(1/\varepsilon\right)$, which requires $\delta = \mathcal{O}\left(\mathrm{e}^{-c/\varepsilon}\right)$ for some $c > 0$.

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When $\sigma_\pm, \tau_\pm = \mathcal{O}(1/\varepsilon)$, the two-term approximation $\phi_0 + \varepsilon\phi_1$ is continuous across Γ and satisfies

$$\left[\frac{\partial \phi_0}{\partial n} + \varepsilon \frac{\partial \phi_1}{\partial n}\right] = \alpha\left(\phi_0 + \varepsilon\phi_1\right) \qquad \text{on } \Gamma,$$

$$\alpha = \frac{2\pi}{\varepsilon\left(\log\left(1/(2\pi\delta)\right) + a_0\right)} = \frac{2\pi M/|\Gamma|}{\log\left(|\Gamma|/(2\pi r M)\right) + a_0}.$$

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- if $\delta \ll \mathcal{O}\left(\mathrm{e}^{-c/\varepsilon}\right)$ for every $c > 0$ (very thin wires), then there is no shielding.
- as $\delta \to \delta_\infty = \mathrm{e}^{-a_0}/(2\pi)$, $\alpha \to \infty$ and the approximation breaks down (for circular wires $\delta_\infty = 1/(2\pi) \approx 0.16 < \delta_* = 1/2$).

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Thick wires ($\delta = \mathcal{O}(1)$). If δ is strictly $\mathcal{O}(1)$ then $\sigma_\pm, \tau_\pm$ are $\mathcal{O}(1)$. Hence

$$\phi_0^+ = \phi_0^- = 0 \qquad \text{on } \Gamma.$$

So the leading order solution is as for a solid (Dirichlet) shell.
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In particular, $\phi_0^- = 0$ in Ω_-, provided k is not a resonant wavenumber.

At $\mathcal{O}(\varepsilon)$,

$$\phi_1^+ = \sigma_+\frac{\partial\phi_0^+}{\partial n} - \tau_-\frac{\partial\phi_0^-}{\partial n} \qquad \text{on } \Gamma,$$
$$\phi_1^- = \tau_+\frac{\partial\phi_0^+}{\partial n} - \sigma_-\frac{\partial\phi_0^-}{\partial n} \qquad \text{on } \Gamma.$$

So $\phi_1^\pm$ satisfy Dirichlet problems, with data coming from the $\mathcal{O}(1)$ solutions.

Resonance

Near a resonant wavenumber k^* of the solid Dirichlet shell, the "thick wire" approximation breaks down. We expect a large field in Ω_-, so try

$$\phi^-(x,y) = \frac{1}{\varepsilon}\phi^-_{-1}(x,y) + \phi^-_0(x,y) + \varepsilon\phi^-_1(x,y) + \mathcal{O}\left(\varepsilon^2\right) \qquad \text{in } \Omega_-$$

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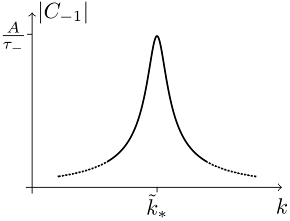
Then

$$\phi^-_{-1} = C_{-1}\psi^*,$$

where ψ^* is the eigenmode corresponding to k^* (more generally a superposition), and solvability conditions for ϕ^-_0 and ϕ^-_1 reveal that

$$|C_{-1}| = \frac{A}{\tau_-}\left(1 + \left(\frac{k - \tilde{k}^*}{\varepsilon^2\tau_-\tau_+ a}\right)^2\right)^{-1/2}$$



where $\tilde{k}^* = k^* + \varepsilon\tilde{k}^*_1 + \varepsilon^2\tilde{k}^*_2$ and

the constant	A	a	$\tilde{k}^*_1$	$\tilde{k}^*_2$
depends on	Γ	Γ	Γ, K	Γ, K

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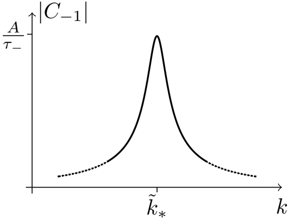

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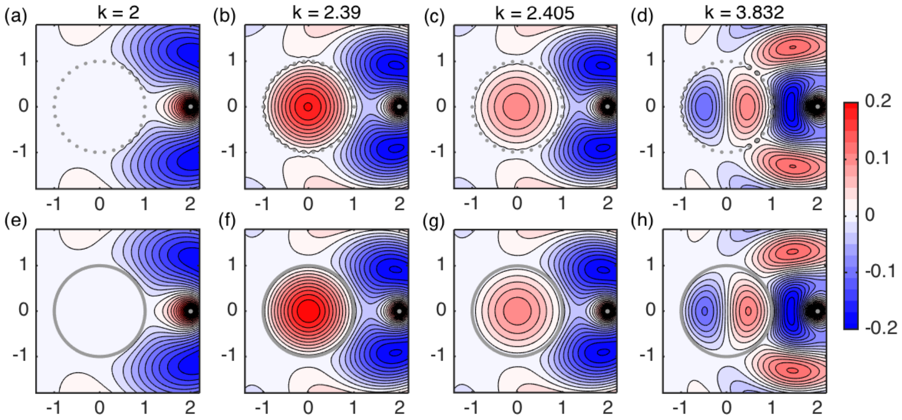
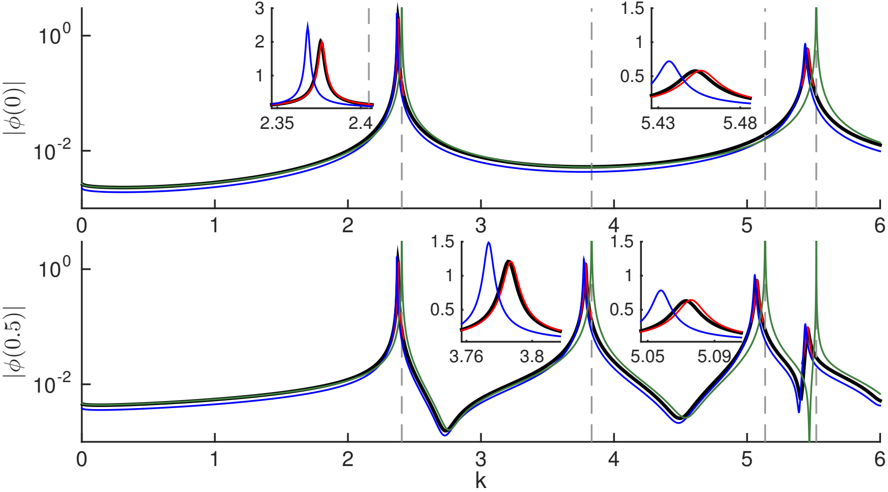
So the resonant response occurs in a range of wavenumbers of width $\mathcal{O}\left(\tau_+\tau_-\varepsilon^2\right)$ around $k = \tilde{k}^*$ with maximum amplitude $\mathcal{O}\left(\frac{1}{\tau_-\varepsilon}\right)$

Asymptotics vs numerics



Parameters: $M = 30$, $\delta = 0.1$, $(\varepsilon \approx 0.21)$, $z_0 = 2$.

Asymptotics vs numerics

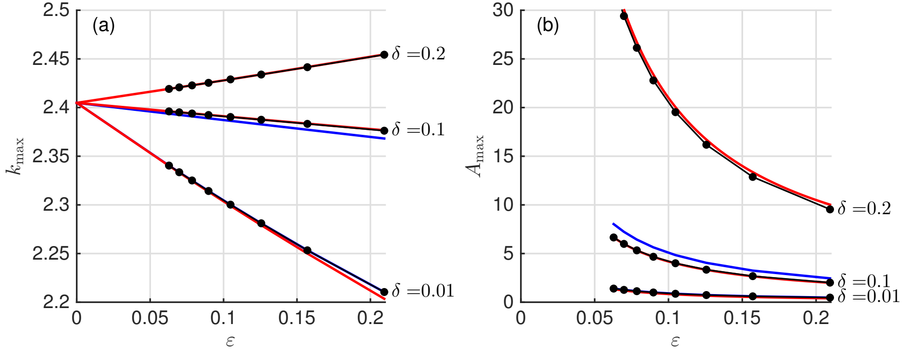


Key: numerical, thin wire homog, thick-wire homog, thick-wire resonant homog

Parameters: $M = 30$, $\delta = 0.1$, $(\varepsilon \approx 0.21)$, $z_0 = 2$.

Asymptotics vs numerics

Location and amplitude of near-resonant peak:



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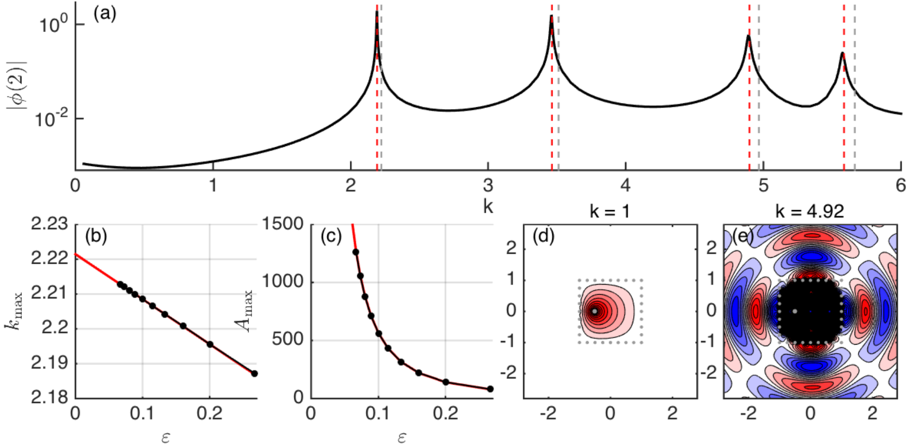
NB: thin wire homog not applicable for $\delta = 0.2$ (so not plotted)

Application to non-smooth Γ



Key: numerical, thick-wire resonant homog

Parameters: $M = 32$, $\delta = 0.1$, $(\varepsilon = 0.125)$, $z_0 = -0.5$.

(cf. work of Delourme/Schmidt/Semin on related problems)

Thank you!

For more details:

Chapman, Hewett and Trefethen, *Mathematics of the Faraday cage*, SIAM Review, 57(3), 398-417, 2015

Hewett and Hewitt, *Homogenized boundary conditions and resonance effects in Faraday cages*, Proc. Roy. Soc. A, 472, 2189, 2016

(links and preprints at: `www.ucl.ac.uk/∼ucahdhe`)