Existence of a translating pair of oppositely rotating vortex batch

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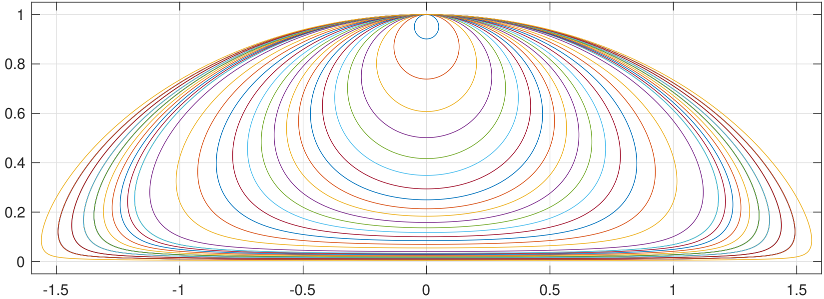
Vortex Patch Pair

A canonical problem in 2-D vortex dynamics is a steady translating pair of oppositely rotating vortex patch. These were computed numerically (Pierrehumbert (1981)). Family of vortex shapes are shown below:



Shape of upper-half vortex for different sizes

- Until now, mathematical analysis limited to small vortices.
  Objective: Extend this to any sized vortices.

Equation to be satisfied

For a given choice of length scale and vorticity ω, determine a simple differentiable closed curve ∂D_1, given by $Z : [0, 2\pi] \to \mathbb{C}$, and corresponding translation velocity U so that

$$\Im \left\{ \frac{dZ}{d\nu} \frac{dw}{dz} \right\} = 0 \ , \ \text{on } \partial D_1 \tag{1}$$

where

$$\frac{dw}{dz} = -U + \frac{1}{4\pi} \oint_{\partial D_1} \left(\frac{\overline{z - z'}}{z - z'} + \frac{\overline{z + z'}}{z + z'} \right) dz' \, . \tag{2}$$

Methodology Based on Quasi-Solution idea

Consider a nonlinear problem, written abstractly as $\mathcal{N}[v] = 0$.
Initial/Boundary conditions appear as auxiliary conditions on v.
Suppose, one can find v_0 which satisfies initial/BC approximately and $\mathcal{N}[v_0] = R$ is small. Then $E = v - v_0$ satisfies

$$\mathcal{L}E = -R - \mathcal{N}_1[E] \ ,$$

where $\mathcal{L} = \mathcal{N}_u$, $\mathcal{N}_1[E] = \mathcal{N}[v_0 + E] - \mathcal{N}[v_0] - \mathcal{L}E$
If $\mathcal{L}$ has bounded inverse in some appropriate space, and nonlinearity $\mathcal{N}_1$ is regular, then E satisfies the weakly nonlinear equation:

$$E = E_0 - \mathcal{L}^{-1}R - \mathcal{L}^{-1}\mathcal{N}_1[E]$$

where E_0 solves $\mathcal{L}E_0 = 0$ and satisfies small IC/BC. Note $v = v_0 + E$ will then solve $\mathcal{N}[v] = 0$ and desired IC/BC.

Remarks

Inversion of this type needed in bounding $|u - u_0|$ in a problem of the type: $\mathcal{N}[u;\epsilon] = 0$ where u_0 exactly solves $\mathcal{N}[u_0; 0] = 0$

Previously, this idea used to determine errors in numerical solution to elliptic PDEs(Nakao *et al*, (2005); determine existence of Stokes Water Wave (Fraenkel, '07) using a rough u_0. Computer assisted proof by Kobayashi ('04). Computer assisted proof have been used in many PDE/ODE contexts

Not recognized until recently is that accurate analytical approximate solution u_0 (called quasi-solution) representation possible with rigorous error estimate.

Recent work in this direction: (Costin, Huang, Schlag, 2012), (Costin, Huang, T., 2012), (Costin, T., 2013), (Costin, Kim, T. 2014),(T. 2013), Costin, Doninger & Glogic (2016), (Adali, T., 2017), in problems arising in NLS, Proof of Dubrovin Conjecture for P-1, Blasius similarity solution, water waves, stability of blow up in wave maps, and in Hele-Shaw problem.

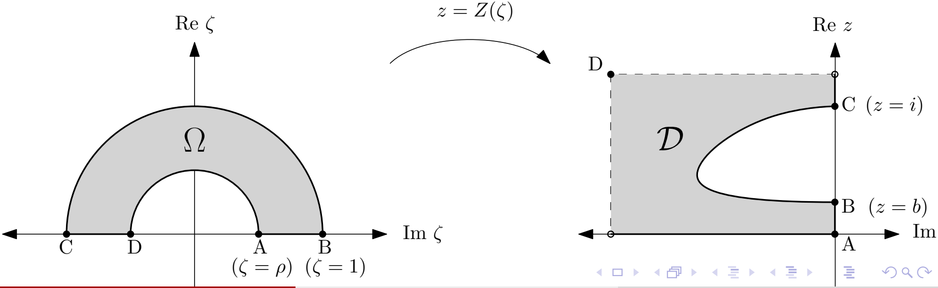
Challenges for QS method in translating vortex pair.

1. The equation as it stands $\Im\left(\frac{dZ}{d\nu}\frac{dw}{dz}\right) = 0$ on the face of it looks nonlinear in $Z'(\nu)$ since complex velocity $\frac{dw}{dz}$ also involves $Z'(\nu)$.
2. The challenge is to find an equivalent formulation where the Frechet derivative $\mathcal{L}$ of the nonlinear operator at some approximate solution can be shown to have bounded inverse with the bound explicitly determined.
3. The choice of space has to be made so that all the nonlinear terms, which also occurs in nonlocal manner, can be controlled through some Banach Algebra property.
4. It is *a priori* unclear how to use the above equation to analytically determine velocity U for given separation of vortex centroids and choice of length and vorticity scales.

Conformal Mapping Representation

Assuming fore-and-aft symmetry, consider conformal map $z = Z(\zeta)$ from domain $\Omega := \{\zeta : \rho < |\zeta| < 1 \ , \ \Im\zeta > 0\}$ to third quadrant region $\mathcal{D}$

$$\begin{cases} \boxed{\mathsf{A}} : & \zeta = +\rho \text{ is mapped to } z = 0, \\ \boxed{\mathsf{B}} : & \zeta = +1 \text{ is mapped to } z = b \text{ for some } b \in (0,1)i, \\ \boxed{\mathsf{C}} : & \zeta = -1 \text{ is mapped to } z = i, \text{ and} \\ \boxed{\mathsf{D}} : & \zeta = -\rho \text{ is mapped to } z = \infty. \end{cases} \tag{3}$$



Useful properties in conformal mapping formulation

$$Z(\zeta) = ia_0 \left\{ \frac{\zeta - \rho}{\zeta + \rho} + \alpha(\zeta) - \alpha\left(\rho^2 \zeta^{-1}\right) \right\} \ ,$$

$$\alpha(\zeta) = \sum_{k=1}^{\infty} a_k \zeta^k \ , \ a_0 \in \mathbb{R}^+ \ , \ a_k \in \mathbb{R}$$

We choose choose $a_0 = 1$, instead of enforcing $\Im Z(-1) = 1$. Treat $\rho \in (0,1)$ as a parmeter, instead of distance between centroids of vortices. $\frac{dw}{dz} - U = O(z^{-2})$ as $z \to \infty$ implies $\frac{dw}{dz} - U$ has a double zero at $\zeta = -\rho$.

$$\left(\frac{dw}{dz} - U \right) \frac{dZ}{d\zeta}$$

is therefore singularity free in $\rho \leq |\zeta| < 1$. Also $i\zeta Z'(\zeta) + \mathcal{J}(\zeta)$ is singularity free in $\rho \leq |\zeta| < 1$ and real on $|\zeta| = \rho, 1$, where

$$\mathcal{J}(\zeta) = 2\zeta \sum_{k=1}^{\infty} \left(\frac{\rho^{2k-1}}{(\zeta + \rho^{2k-1})^2} + \frac{\rho^{2k-1}}{(1 + \rho^{2k-1}\zeta)^2} \right)$$

Alternate Formulation of the problem

$$\text{Recall } Z(\zeta) = i \left\{ \frac{\zeta - \rho}{\zeta + \rho} + \sum_{k=1}^{\infty} a_k \left(\zeta^k - \frac{\rho^{2k}}{\zeta^k} \right) \right\} \text{, where } a_k \in \mathbb{R}$$

Problem: Determine univalent Z in $\rho \leq |\zeta| < 1$ in the above form, with Z' existing on $|\zeta| = 1$, such that

$$\left(U + \mathcal{A}[Z](\zeta) \right) i\zeta Z' \zeta + U \mathcal{J}(\zeta) = \mathcal{C}[Z] \,, \quad \text{where}$$

$$\mathcal{A}[Z](\zeta) = \frac{1}{4\pi} \oint_{|\zeta'|=1} \left(\overline{\frac{Z(\zeta) - Z(\zeta')}{Z(\zeta) - Z(\zeta')}} + \overline{\frac{Z(\zeta) + Z(\zeta')}{Z(\zeta) + Z(\zeta')}} \right) Z_\zeta(\zeta') \, d\zeta' \; , . \quad (4)$$

$$\text{and } \mathcal{C}[Z] = \frac{1}{2\pi} \oint_{|\zeta'|=1} \mathcal{A}[Z](\zeta) Z'(\zeta) d\zeta$$

We seek solution where $k^2 a_k \in l^2(\mathbb{Z}^+)$.

Formulation on $|\zeta| = 1$ suitable quasi-solution strategy

$$\text{On } \zeta = e^{i\nu}, \quad \frac{dZ}{d\nu} = -Q\left(e^{i\nu}\right)\left[-\mathcal{C}[Z] + U\mathcal{J}\left(e^{i\nu}\right)\right] \ , \tag{5}$$

where reciprocal complex velocity

$$Q(\zeta) = \left(U + \mathcal{A}[Z](\zeta)\right)^{-1}$$

With $Z(\zeta) = F_0(\zeta) + G(\zeta)$, $U = U_0 + u$, define

$$E(\nu) = \Re G(e^{i\nu}) = \sum_{k=1}^{\infty} g_k (1 + \rho^{2k}) \sin(k\nu) \text{ where } g_k \in \mathbb{R} \ .$$

Determine $E \in \mathcal{S}^2$ in the space of real valued odd 2π-periodic function with norm given by

$$\|E\|^2 = \sum_{k=1}^{\infty} k^4 g_k^2 < \infty \ \text{ such that}$$

$$\mathcal{L}E := E'(\nu) + \mathcal{T}[E](\nu) = -\left(\mathcal{R}_0(\nu) + u\mathcal{R}_1(\nu) + u\mathcal{L}_1[E](\nu) + \mathcal{M}[E](\nu)\right)$$

Linear Terms

$$\mathcal{T}[E](\nu) = \Re\left\{ Q_0(e^{i\nu})\left[ie^{i\nu}F_0'(e^{i\nu})A_1[G](e^{i\nu}) - c_1[G]\right]\right\} \ , \qquad (6)$$

where c_1 linear part of functional $\mathcal{C}[G]$, $Q_0(\zeta) = (A_0[F_0](\zeta) + U_0)^{-1}$,

$$A_0(\zeta) = \frac{1}{4\pi}\oint_{|\zeta'|=1}\left(\frac{\overline{F_0(\zeta) - F_0(\zeta')}}{F_0(\zeta) - F_0(\zeta')} + \frac{\overline{F_0(\zeta) + F_0(\zeta')}}{F_0(\zeta) + F_0(\zeta')}\right) F_0'(\zeta')\, d\zeta' \ , \qquad (7)$$

$$A_[G](\zeta) = A_{1,1}(\zeta) + A_{1,2}(\zeta) \ , \qquad (8)$$

where

$$A_{1,1} = \frac{1}{4\pi}\oint_{|\zeta'|=1}\left(\frac{\overline{F_0(\zeta) - F_0(\zeta')}}{F_0(\zeta) - F_0(\zeta')} + \frac{\overline{F_0(\zeta) + F_0(\zeta')}}{F_0(\zeta) + F_0(\zeta')}\right) G'(\zeta')\, d\zeta' \qquad (9)$$

$$+ \frac{1}{4\pi}\oint_{|\zeta'|=1}\left(\frac{\overline{G(\zeta) - G(\zeta')}}{F_0(\zeta) - F_0(\zeta')} + \frac{\overline{F_0(\zeta) + F_0(\zeta')}}{F_0(\zeta) + F_0(\zeta')}\right) F_0'(\zeta')\, d\zeta' \qquad (10)$$

More definition

$$A_{1,2} = -\frac{1}{4\pi}\oint_{|\zeta'|=1}\left(\frac{\overline{F_0(\zeta)-F_0(\zeta')}}{\left(F_0(\zeta)-F_0(\zeta')\right)^2}\right)\left(G_0(\zeta)-G_0(\zeta')\right)F_0'(\zeta')\,d\zeta' \tag{11}$$

$$-\frac{1}{4\pi}\oint_{|\zeta'|=1}\left(\frac{\overline{F_0(\zeta)+F_0(\zeta')}}{\left(F_0(\zeta)+F_0(\zeta')\right)^2}\right)\left(G(\zeta)+G(\zeta')\right)F_0'(\zeta')\,d\zeta' \tag{12}$$

We define the projection P so that if $h(\nu)=\sum_{k=0}^{\infty} h_k\cos(k\nu)$

$$P[h](\nu)=\sum_{k=1}^{\infty} h_k\cos(k\nu)$$

Overall Strategy

1. For given $u \in [-\epsilon, \epsilon] =: I$, prove

$$\tilde{\mathcal{L}}E := P\mathcal{L}[E] = P\left[\mathcal{R}_0 + u\mathcal{R}_1 + \mathcal{M}[E] + u\mathcal{L}_1[E]\right] =: \tilde{\mathcal{R}} \qquad (13)$$

   has a unique solution for $E \in \mathcal{S}^2$ in some small ball.

2. Show that there exists unique $u \in [-\epsilon, \epsilon]$ so that the remaining scalar equation

$$(I - P)\mathcal{L}[E] = (I - P)\left[\mathcal{R}_0 + u\mathcal{R}_1 + \mathcal{M}[E] + u\mathcal{L}_1[E]\right] \qquad (14)$$

   is satisfied.

3. Using properties of explicit quasi-solution F_0 and the smallness of G (note it is determined once E is known), prove that $Z(\zeta) = F_0(\zeta) + G(\zeta)$ is univalent in the domain Ω.

Strategy for proving solution for $\tilde{\mathcal{L}}E = \tilde{\mathcal{R}}$

1. Prove that residual $\|P\mathcal{R}_0\|_1$ is small in the space $\mathcal{C}^1$, *i.e.* space of real functions with cosine series with zero average: $\sum_{k=1}^{\infty} b_k \cos(k\nu)$ with norm $\left(\sum_{k=1} k^2 b_k^2\right)^{1/2}$.
2. Prove that $\tilde{\mathcal{L}}^{-1} : \mathcal{C}^1 \to \mathcal{S}^2$ exists and is uniformly bounded for any $u \in I$.
3. Using Banach algebra properties of the chosen norm showing nonlinear term $\|\mathcal{M}[E]\|_1 \leq C\|E\|_2^2$ for some constant C which is explicitly estimated.

With these steps, we can complete the prove that vortex patch solution exist near the quasi-solution in the same manner as for a general nonlinear problem indicated at the outset. We also calculate bounds on $\|\partial_u E\|$.

Main Result

Theorem

For $\rho = 0.447$, there exists a unique solution (U, Z) to the vortex patch problem in a small neighborhood of (U_0, F_0) where

$$U_0 = \frac{197586005079}{922177195000}, \quad F_0(\zeta) = i\left[\frac{\zeta-\rho}{\zeta+\rho} + \alpha(\zeta) - \alpha(\rho^2/\zeta)\right], \qquad (15)$$

where $\alpha(\zeta) = \sum_1^{20} a_k \zeta^k$ as given later. More precisely,

$$\|Z - F_0\| \leq 5.75 \times 10^{-5} \quad \text{and} \quad \Big|U - U_0\Big| \leq 4 \times 10^{-6}. \qquad (16)$$

Furthermore, this solution F is univalent in the annular domain Ω.

Coefficients $(a_1, a_2, \cdots, a_{20})$ are given by

$$\begin{aligned}\Big(&\frac{15142429}{90818483}, -\frac{8719714}{209166067}, \frac{5283256}{411082005}, -\frac{1911385}{440092704}, \\ &\frac{1522929}{980119345}, -\frac{625524}{1082029691}, \frac{412705}{1865723458}, -\frac{294493}{3407668220}, \\ &\frac{103448}{3015451213}, -\frac{237097}{17192503397}, \frac{61323}{10949670290}, -\frac{51127}{22291912927}, \\ &\frac{51059}{53979652574}, -\frac{13381}{34096042921}, \frac{10159}{62069985178}, -\frac{4450}{64901421047}, \\ &\frac{5433}{188400992941}, -\frac{1564}{128502607763}, \frac{529}{102661979487}, -\frac{3498}{1598964511447}\Big).\end{aligned} \tag{17}$$

Conclusion

1. We have shown how the nonlocal nonlinear problem of a steadily translating pair of oppositely rotating vortices can be analyzed using a quasi-solution method.
2. One of the important steps was to use analytic properties of conformal map and complex velocity $\frac{dw}{dz}$ to prove that $\Im\left\{\frac{dZ}{d\nu}\left(U+\mathcal{A}[Z]\right)\right\}=0$ is equivalent to
$$\frac{dZ}{d\nu}=\frac{\mathcal{C}-\mathcal{J}(\zeta)}{U+\mathcal{A}[Z]}$$
for some known function $\mathcal{J}$ and for some real functional $\mathcal{C}$.
3. The method has no *a priori* restriction on centroid distance parameter $\rho \in (0,1)$. When ρ gets closer to 1, the solution representation gets more complicated.
4. We showed that the space where power series coefficients $\{a_k\}_{k=1}^{\infty}$ satisfies $k^2 a_k \in l^2(\mathbb{Z}^+)$ is suitable for controlling both nonlinear and nonlocal terms.

Finding bounds on $\tilde{\mathcal{L}}^{-1} : \mathcal{C}_1 \to \mathcal{S}_2$

We note first $\tilde{\mathcal{L}}E$ contains two terms, the first is simply $E'(\nu)$ and the second $\mathcal{T}E$ involves Q_0, the reciprocal of the complex velocity $U_0 + \mathcal{A}[F_0]$, based on quasi-solution (U_0, F_0) and A_1 whose derivative with respect to ν contains terms such as

$$\frac{i}{4\pi} \oint_{|\zeta'|=1} \frac{\overline{\zeta' G'(\zeta')}}{F_0(\zeta) - F_0(\zeta')} F_0'(\zeta') d\zeta' \qquad (18)$$

Notice that the large Fourier component of this contribution is dominated by Hilbert Transform type component and we know that L^2 norm of a Hilbert transform is the same as the the original function. Since Q_0 is based on an explicit function F_0 determining bound on Q_0 and Q_0' is relatively simple. This allows us to estimate

$$\|\frac{d}{d\nu} P\mathcal{T}[E]\| \leq C_0 \|E\|_{\mathcal{S}_1}$$