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A Theoretical Study of Low-Reynolds Number Swimming near Corners

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Structure of the Talk

- Introduction: Outline how we can study 2D Stokes flow using Complex Variables, Introduce the Crowdy-Or model swimmer, Glance over the work that has been carried out so far, Motivation to study a more general problem.

- Solving the Problem: Outline the problem and aims, Briefly discuss corner singularities, Key Methodologies.

- Results: Show how the method treated the corner singularities, Full Dynamical System governing the swimmer, Equilibria of the system - Scattering and Trapping of microorganisms.

Complex formulation of Stokes Flow

The Stokes equations in two dimensions (the equations governing viscous flow in 2D) are

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0, \tag{1}$$

where ∇^2 is the usual Laplacian operator in 2D, $\mathbf{u} = (u, v)$ is the velocity field and p the fluid pressure. The flow is incompressible and 2D, so we can introduce a streamfunction ψ:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{2}$$

Taking the curl of the first equation in (1) gives

$$\nabla^2 \omega = 0, \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{3}$$

But from (2) and (3) we can also deduce the relationship

$$\nabla^2 \psi = -\omega. \tag{4}$$

Together (3) and (4) imply

$$\nabla^4 \psi = 0. \tag{5}$$

We now change to the complex variables

$$z = x + iy, \quad \bar{z} = x - iy, \tag{6}$$

with which the biharmonic equation (5) becomes

$$\frac{\partial^4 \psi}{\partial \bar{z}^2 \partial z^2} = 0. \tag{7}$$

This can be integrated to provide the solution for the streamfunction in terms of two analytic functions:

$$\psi = \mathrm{Im}\{\bar{z} f(z) + g(z)\} \tag{8}$$

We can then recover all the physical quantities of interest from

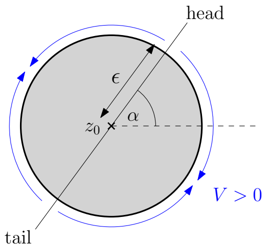
$$\boxed{p - i\omega = 4f'(z),} \tag{9}$$

$$\boxed{u - iv = -\overline{f(z)} + \bar{z} f'(z) + g'(z).} \tag{10}$$

The Crowdy-Or treadmilling Swimmer

Circular swimmer, radius ϵ, position z_0. α is the head-tail orientation angle. On the swimmers surface a tangential slip velocity is imposed of the form

$$2V \sin 2(\phi - \alpha). \tag{11}$$



The swimmer corresponds to a superposition of a point stresslet and a source quadrupole at z_0.

The Goursat functions for the swimmer take the form

$$f(z) = \frac{\mu}{z - z_0} + \text{locally analytic}, \tag{12}$$

$$g'(z) = \frac{\mu \overline{z_0}}{(z - z_0)^2} + \frac{2\mu\epsilon^2}{(z - z_0)^3} + \text{locally analytic}, \tag{13}$$

where

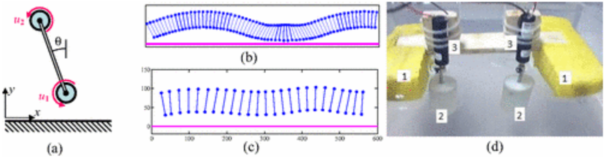
$$\mu = \epsilon V e^{2i\alpha}. \tag{14}$$

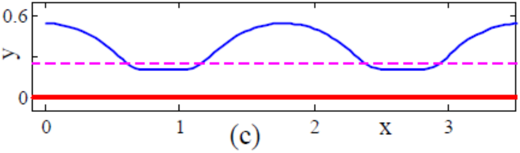
Important Point

This formulation for the model swimmer using point singularities is **exact** in free space, it becomes an **approximation** when we put the swimmer in some other geometry where walls are present.

What has been done so far

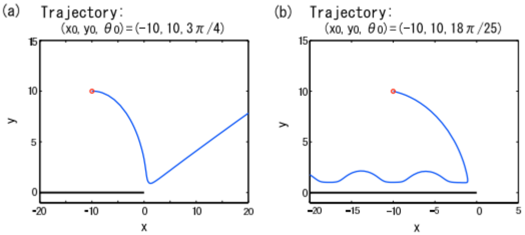
Crowdy, Or (2010)



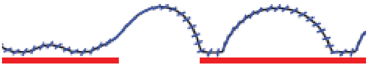


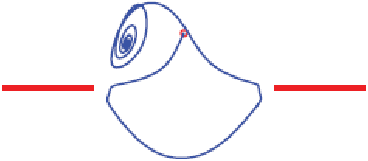
This simple singularity model captures the behavior of the rod swimmer.

Obuse, Thiffeault (2012)



Crowdy, Samson (2011)

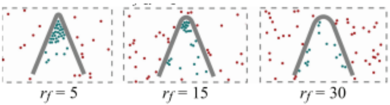


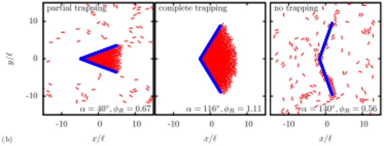


Motivation to study the general Wedge

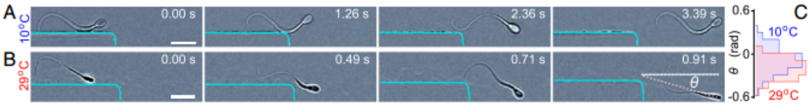
- The Crowdy-Or model provides a simple model for calculating qualitatively accurate swimmer behavior.
- The wedge angles of π and 2π have been solved already by Crowdy, Or (2010) and Obuse, Thiffeault (2012).

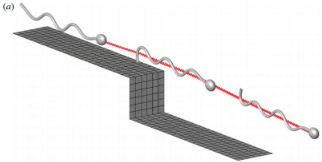
There has been much work on the trapping of microorganisms in wedge shaped geometries, to name just a few: Kaiser, Wensink, Lowen (2012), and with Popowa (2013). Guidobaldi et al. (2014) investigated sperm cell trapping.





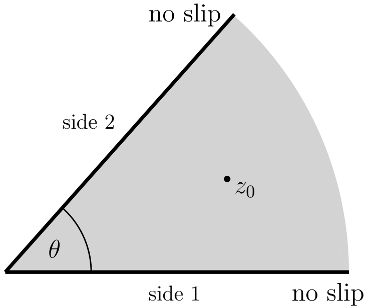
As well as microorganism scattering in 'open' wedges: Kantsler, Dunkel, Polin, Goldstein (2013). Montenegro-Johnson, Gadelha, Smith (2015).





Outline of the Problem

Consider a swimmer at z_0 in Stokes flow in a wedge of angle $\theta \in (0, 2\pi]$. The walls here are solid and we will use no slip boundary conditions.



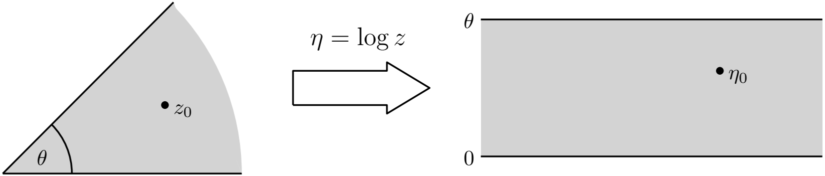
Our aim is to find the dynamical system governing the motion of the swimmer inside the wedge.

Why must we be careful about how we deal with the corner point

- For an arbitrary θ, we cannot use a 'method of images approach' as in the π and 2π cases.
- We must be careful to deal properly with the singularity structure near to the corner point. Dean and Montagnon (1949) and Moffatt (1964) have found the local solution behavior near the corner, we must ensure our method takes account of this.
- For wedge angles suffieciently small, Moffatt eddies are present.

Conformal Mapping

We apply the conformal map $\eta = \log z$, to enable us to work in the channel geometry rather than the wedge geometry.



This proves to be very beneficial; we can use a transform method (to be outlined soon) in a convenient geometry and we can express the forcing functions, the $f(z)$ and $g'(z)$ for the swimmer, in an important way too.

Representing the Goursat functions

We put

$$F(\eta) = f(z), \quad G(\eta) = g'(z), \tag{15}$$

and then in the η-plane we decompose as

$$F(\eta) = F_s(\eta) + \hat{F}(\eta), \quad G(\eta) = G_s(\eta) + \hat{G}(\eta), \tag{16}$$

where we choose

$$F_s(\eta) = \frac{\mu}{z_0}\left[\frac{\pi}{2\theta}\coth\left(\frac{\pi(\eta-\eta_0)}{2\theta}\right)\right] \quad \sim \quad \frac{A}{\eta-\eta_0}, \text{ near } \eta = \eta_0 \tag{17}$$

$$G_s(\eta) = \frac{\mu\overline{z_0}}{z_0^2}\left[\left(\frac{\pi}{2\theta}\right)^2 \operatorname{cosech}^2\left(\frac{\pi(\eta-\eta_0)}{2\theta}\right)\right] - \frac{\mu\overline{z_0}}{z_0^2}\left[\frac{\pi}{2\theta}\coth\left(\frac{\pi(\eta-\eta_0)}{2\theta}\right)\right]$$
$$+\frac{\overline{\mu}}{\overline{z_0}}\left[\frac{\pi}{2\theta}\coth\left(\frac{\pi(\eta-\overline{\eta_0})}{2\theta}\right)\right] + \frac{\mu\overline{z_0}}{z_0^2}\left[\frac{\pi}{2\theta}\coth\left(\frac{\pi(\eta-\overline{\eta_0})}{2\theta}\right)\right].$$

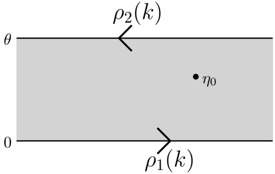
Fourier Transforms in a channel - Crowdy, Davis (2013)

We can seek the remaining analytic functions $\hat{F}(\eta)$ and $\hat{G}(\eta)$ by taking Fourier transforms along the channel walls, for $\hat{F}(\eta)$:

$$\hat{F}(\eta) = \frac{1}{2\pi}\int_0^{\infty} \rho_1(k)e^{ik\eta}\mathrm{d}k + \frac{1}{2\pi}\int_0^{-\infty} \rho_2(k)e^{ik\eta}\mathrm{d}k, \qquad (18)$$

where

$$\rho_1(k) = \int_{-\infty}^{\infty} \hat{F}(\eta)e^{-ik\eta}\mathrm{d}\eta, \quad \rho_2(k) = \int_{i\theta+\infty}^{i\theta-\infty} \hat{F}(\eta)e^{-ik\eta}\mathrm{d}\eta. \qquad (19)$$



Global relation: $\rho_1(k) + \rho_2(k) = 0, \quad k \in \mathbb{R}$

Spectral Function $\rho_1(k)$

- Utilising the no-slip boundary conditions and the global relation it is possible to find the spectral functions explicitly. Once we have $\rho_1(k)$ it is easy to get $\rho_2(k)$ via the global relation.
- We find

$$\rho_1(k) = \frac{(e^{2k\theta}-1)\overline{R}(-k) - ik(1-e^{2i\theta})R(k)}{(e^{2k\theta}-1)(e^{-2k\theta}-1) + k^2(1-e^{2i\theta})(1-e^{-2i\theta})}.$$

  $R(k)$ is a function we can calculate explicitly via residue calculus, it comes from integrals of the forcing functions along the boundaries.
- Put $\Delta(k) = (e^{2k\theta}-1)(e^{-2k\theta}-1) + k^2(1-e^{2i\theta})(1-e^{-2i\theta})$.

The denominator $\Delta(k)$ can be written as $4(k^2 \sin^2\theta - \sinh^2 k\theta)$, which, when set to zero, is essentially the Moffatt eigencondition. For example:

- $\theta = \pi$:

  $$\sinh^2(k\pi) = 0 \quad \Rightarrow \quad k = im,\ m \in \mathbb{Z} \quad \Rightarrow \quad e^{ik\eta} = z^{ik} = z^{-m}$$

  It's analytic near the corner.

- $\theta = 2\pi$:

  $$\sinh^2(2k\pi) = 0 \quad \Rightarrow \quad k = \frac{im}{2},\ m \in \mathbb{Z} \quad \Rightarrow \quad e^{ik\eta} = z^{ik} = z^{-m/2}$$

  There are square root singularities near the corner.

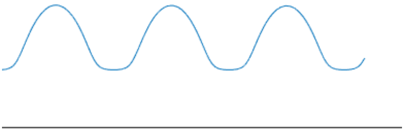
The transform method is fully resolving the local behavior near the corner. For full details: Crowdy, Brzezicki (2017) (submitted).

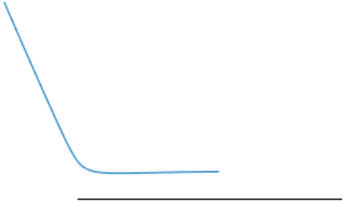
Dynamical system for the Crowdy-Or swimmer in a wedge of angle θ

$$
\begin{aligned}
\frac{\mathrm{d}\overline{z_0}}{\mathrm{d}t} &= -\frac{1}{2}\frac{\overline{\mu}}{\overline{z_0}} + e^{\overline{\eta_0}}\left(\frac{-\mu}{z_0}\right)\left[\frac{1}{12 z_0} + \frac{I_2}{z_0}\right] + \frac{\mu \overline{z_0}}{z_0^2}\left[-\frac{5}{12} + I_2\right] \\
&+ \left[\frac{\overline{\mu}}{\overline{z_0}} + \frac{\mu\overline{z_0}}{z_0^2}\right]\frac{\pi}{2\theta}\coth\left(\frac{\pi(\eta_0 - \overline{\eta_0})}{2\theta}\right) \\
&+ \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[R_1(k)e^{ik\eta_0} + ik\rho_1(k)[e^{\overline{\eta_0}-\eta_0} - 1]e^{ik\eta_0} + \overline{\rho_1}(k)[e^{-ik\eta_0} - e^{-ik\overline{\eta_0}}]\right]\mathrm{d}k \\
&+ \epsilon^2\left\{\frac{\mu}{z_0^3}\left[\frac{3}{4} - 3I_2\right] - \frac{2\mu}{z_0^3}\left[\left(\frac{\pi}{2\theta}\right)^2 \operatorname{cosech}^2\left(\frac{\pi(\eta_0 - \overline{\eta_0})}{2\theta}\right) + \frac{\pi}{2\theta}\coth\left(\frac{\pi(\eta_0-\overline{\eta_0})}{2\theta}\right)\right]\right. \\
&\left. + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\tilde{R}_1(k)e^{ik\eta_0} + ik\tilde{\rho}_1(k)[e^{\overline{\eta_0}-\eta_0} - 1]e^{ik\eta_0} + \overline{\tilde{\rho}_1}(k)[e^{-ik\eta_0} - e^{-ik\overline{\eta_0}}]\right]\mathrm{d}k\right\}, \\
\frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -\frac{1}{2}\mathrm{Im}\left\{-\frac{4\mu}{z_0}\left(\frac{1}{12 z_0} + \frac{I_2}{z_0}\right) + 4e^{-\eta_0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} ik\rho_1(k)e^{ik\eta_0}\mathrm{d}k\right)\right. \\
&\left. + \epsilon^2\left\{4e^{-\eta_0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} ik\tilde{\rho}_1(k)e^{ik\eta_0}\mathrm{d}k\right)\right\}\right\}.
\end{aligned}
$$

Checking Results

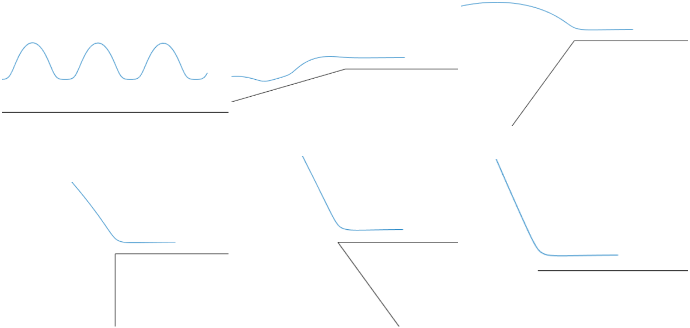
For the cases where the wedge angle θ is π or 2π analytical results have been found. We were able to check our numerical values against the analytic values from these works and our values agree to computer accuracy. We can recreate plots of the sort of behavior found:





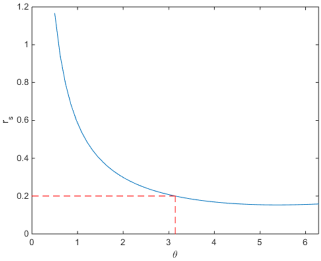
Scattering

- For wedge angles greater than π the swimmer begins to deflect from the corner point, and the precise value of the angle makes little difference for wedges larger than $3\pi/2$.



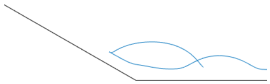
Equilibrium Points

- It is of interest to seek equilibria of the system; points where the RHS of the dynamical system is zero.
- We find an equilibrium always exists along the centreline of the wedge, with orientation angle $\alpha = \theta/2 \pm \pi$.
- The distance of the equilibrium along this centreline, from the corner, is represented by r_s and is plotted against θ below.

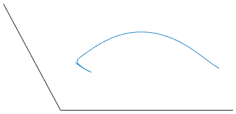


Change of Stability of Equilibrium

- $\theta = 4\pi/5$



- $\theta = 2\pi/3$



It seems that the stability changes from stable to unstable upon increasing θ through about 143°, this is interestingly near the Moffatt angle of 146°.

Summary

- Introduced a simple model swimmer which uses complex analysis methods to provide easily computable results.
- The key methodologies are essentially conformal mapping and Fourier transforms.
- This method provides explicit systems and results, avoiding other time and power consuming numerical methods.
- This extends the work of Crowdy, Or (2010) and Obuse, Thiffeault (2012) to wedges of any angle, where we have found an interesting change in the dynamical behavior of swimmers around angles of 143°, close to the famous Moffatt angle.

Thank you for listening.