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Point Vortex Dynamics on a Toroidal Surface

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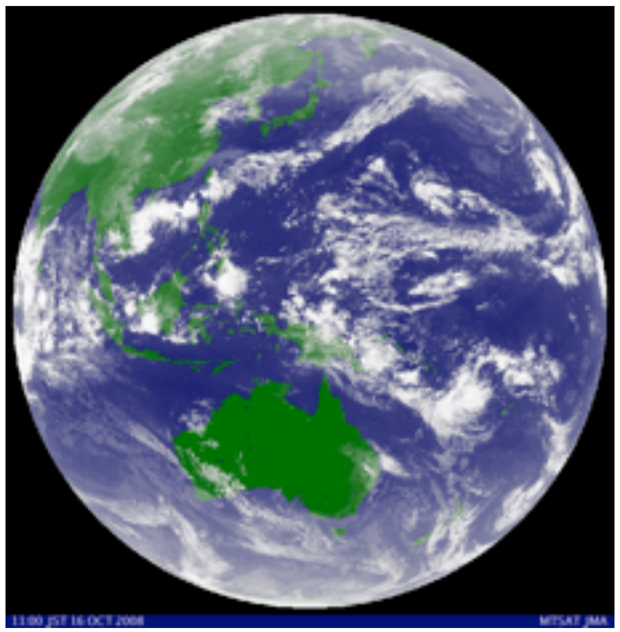
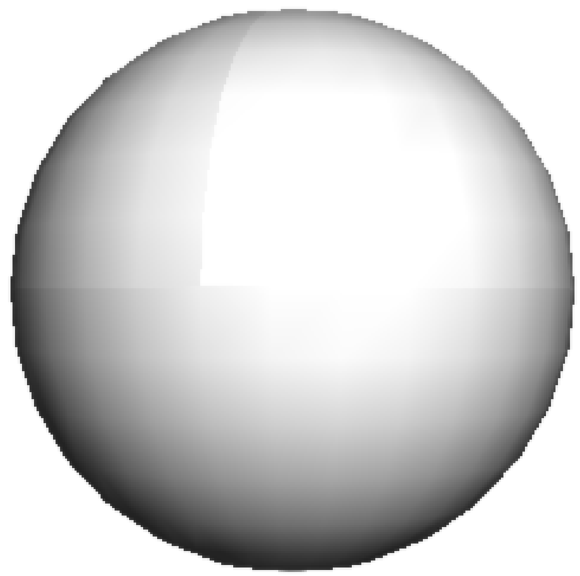
T. Sakajo and Y. Shimizu,
Point vortex interactions on a toroidal surface,
Proc. Roy. Soc. A 472 (2016) 20160271

Vortex flows on curved surfaces

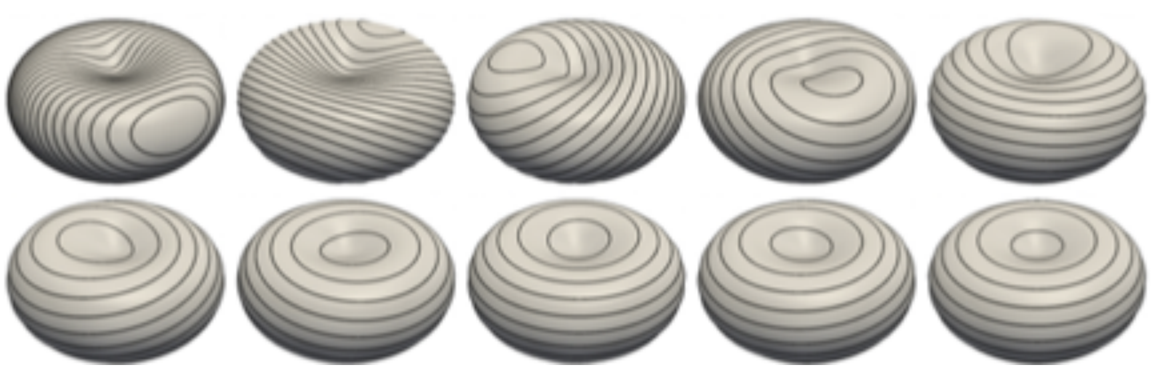
Flows on Earth

They are modeled by flows on a sphere.
Geophysical applications





Flows on Surface of Revolutions



R. Reuther and A. Voigt, A. Voigt et al. 2016

They are applicable to biological flows on membrane

"Exact Solution for Superfluid Film Vortices on a Torus"
A. Corrada-Emmanuel, PRL vol 72, 1994

The problem of describing the velocity field of quantized vortices on a torus ... is used to discuss the behavior of superfluid on porous media.

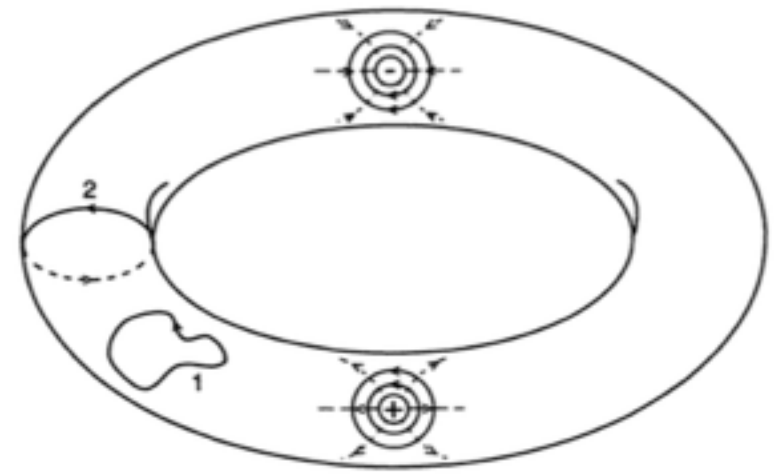


FIG. 1. Torus with two film vortices on its surface.

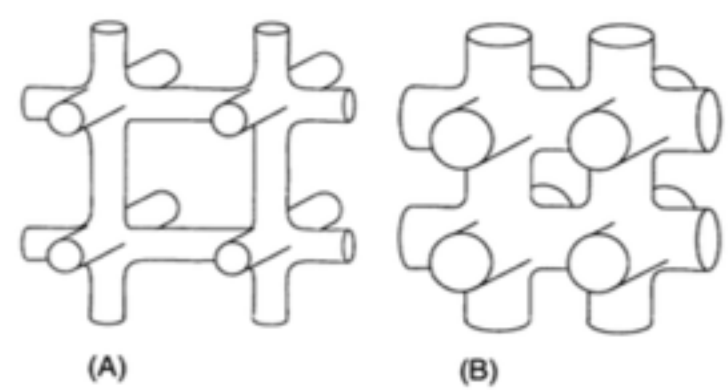
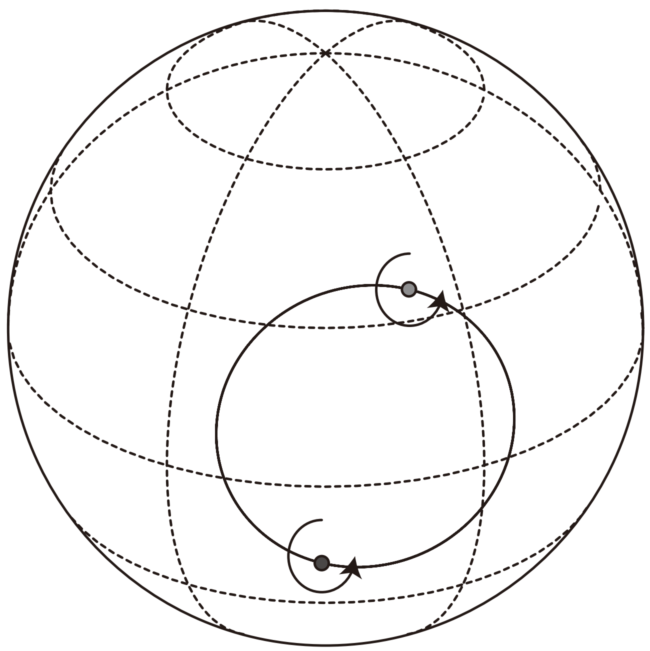


FIG. 3. "Jungle-gym" model of porous materials: (a) Low-p lattice; (b) high-p lattice.

Point vortex interactions

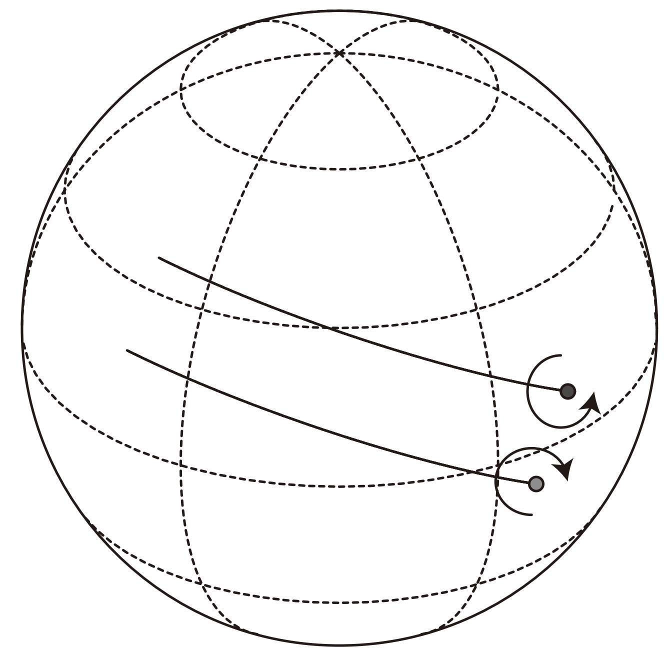
Interaction of two point vortices on a sphere and in unbounded plane without boundary is simple.

Identical vortex pair:
Point vortices with the same strength



The pair is co-rotating

Vortex Dipole:
Point vortices with the opposite strengths



The dipole translates

What happens if they are placed in the surface of a torus, which has non-constant curvature and one non-trivial handle?

Euler flows on 2D Riemannian manifolds

(M, g) M orientable 2D Riemannian manifold g Riemannian metric

Hydrodynamic Green's function (Lin 1941) $G_H(\zeta, \zeta_0)$ $\zeta, \zeta_0 \in M$

$$G_H(\zeta, \zeta_0) = G_H(\zeta_0, \zeta) \quad \text{Reciprocal condition}$$

$$\triangle G_H(\zeta, \zeta_0) = \begin{cases} \delta_{\zeta_0} - \frac{1}{\mathrm{Area}(M)} & \text{if } M \text{ is compact,} \\ \delta_{\zeta_0} & \text{if } M \text{ is non-compact,} \end{cases}$$

$\triangle$ Laplace Beltrami operator δ_{ζ_0} Dirac delta function on $\zeta_0 \in M$

Incompressible and inviscid flows ω vorticity ψ Streamfunction

Poisson equation $-\triangle \psi = \omega$

$$\Longrightarrow \quad \psi(\zeta) = -\int_M G_H(\zeta, \zeta_0)\omega(\zeta_0)d\mu(\zeta_0).$$

μ Riemannian volume form

Velocity field is recovered from the streamfunciton.

Point vortex dynamics

N Point vortices $\zeta_j \in M$ Locations of N point vortices $\Gamma_j \in \mathbb{R}$ Strengths

$$\omega = \sum_{j=1}^{N} \Gamma_j \delta_{\zeta_j} \implies \psi(\zeta) = -\sum_{j=1}^{N} \Gamma_j G_H(\zeta, \zeta_j)$$

Modified streamfunction

$$\psi_m(\zeta_m) = -\sum_{j \neq m}^{N} \Gamma_j G_H(\zeta_m, \zeta_j) - \frac{1}{2}\Gamma_m R(\zeta_m)$$

$$R(\zeta_m) = \lim_{\zeta \to \zeta_m} \left[G_H(\zeta, \zeta_m) - \frac{1}{2\pi} \log d(\zeta, \zeta_m) \right]$$ Robin function

$d(\zeta, \zeta_0)$ distance between $\zeta, \zeta_0 \in M$

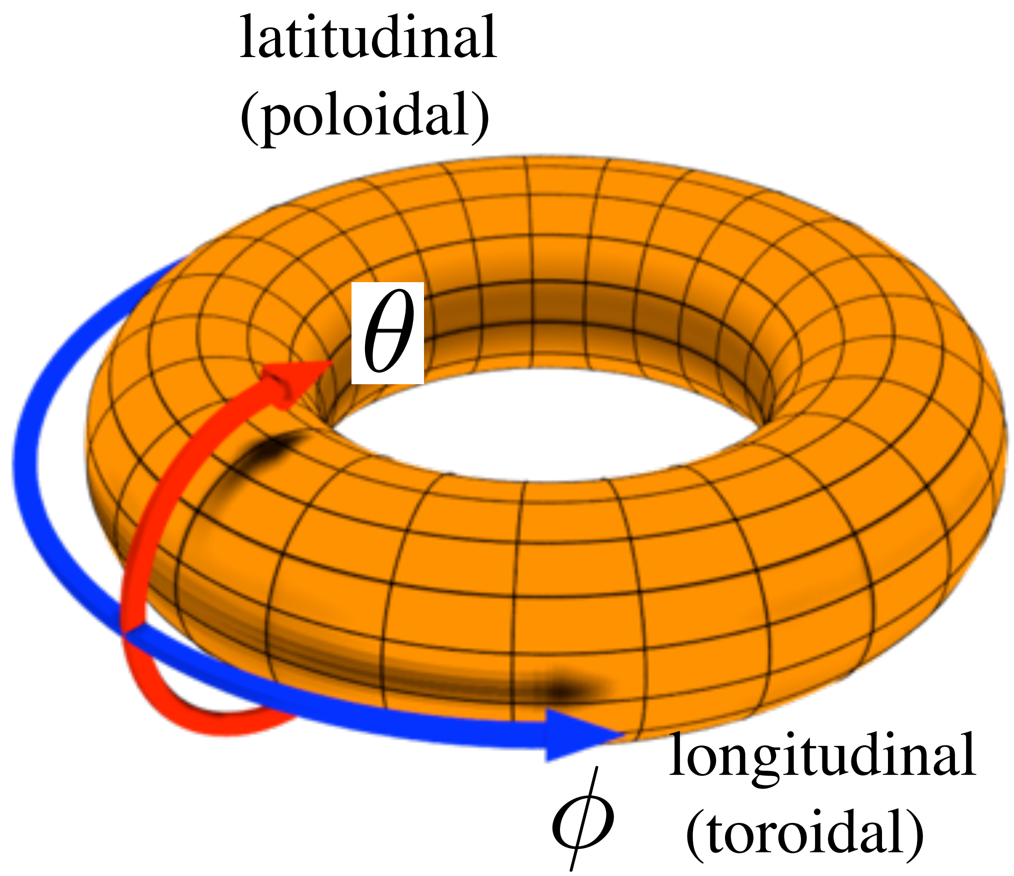
Equations of motion for N point vortices

$$\frac{\mathrm{d}\zeta_m}{\mathrm{d}t} = -2i\lambda^{-2}(\zeta_m, \overline{\zeta}_m) \frac{\partial \psi_m}{\partial \overline{\zeta}_m}$$

$\lambda(\zeta_m, \overline{\zeta}_m)$ conformal factor

A standard toroidal surface

A toroidal surface $\mathbb{T}_{R,r}$



3D Cartesian representation

$$\boxed{\left(R-\sqrt{x^2+y^2}\right)^2+z^2=r^2}$$

R major radius r minor radius

$\alpha = R/r$ aspect ratio

2D surface parametrization

$$(\theta,\phi)\in\mathbb{T}_{R,r}\mapsto((R-r\cos\theta)\cos\phi,(R-r\cos\theta)\sin\phi,r\sin\phi)\in\mathbb{E}^3$$

Complex representation

$$\zeta:(\theta,\phi)\longmapsto \mathrm{e}^{i\phi}\exp\left(-\int_0^\theta\frac{du}{\alpha-\cos u}\right)\equiv \mathrm{e}^{i\phi}\exp(r_c(\theta))$$

Conformal factor

$$\lambda\left(\zeta,\overline{\zeta}\right)=\frac{R-r\cos\theta}{|\zeta|}$$

Distance

$$d(\zeta,\zeta_0)=\lambda(\zeta,\zeta_0)|\zeta-\zeta_0|$$

Hydrodynamic Green's function

Green's function (C. Green & J. Marshall 2012)

$$\boxed{G(\zeta,\zeta_j) = \frac{1}{2\pi}\log|P(\zeta/\zeta_j)| + \varsigma(\eta) + \left(\frac{\log|\zeta_j|}{4\pi^2\mathcal{A}} - \frac{1}{4\pi}\right)\log|\zeta|}$$

$$P(\zeta) = (1-\zeta)\prod_{n\geq 1}(1-\rho^n\zeta)(1-\rho^n\zeta^{-1})$$ Schottky Klein Prime function for the annulus

$$\varsigma(\eta) = \frac{\mathcal{A}}{2\pi^2}\mathrm{Re}\left[\mathrm{Li}_2(c^{-1}\eta)\right] - \frac{1}{2\pi^2\alpha}\log|\eta - c| - \frac{1}{8\pi^2\mathcal{A}}(\log|\zeta|)^2$$

$$\mathrm{Li}_2(\zeta) = -\int_0^\zeta \frac{\log(1-u)}{u}du$$ Dilogarithmic function

$$\rho = \mathrm{e}^{-2\pi\mathcal{A}}$$

$$\mathcal{A} = (\alpha^2-1)^{-1/2}$$

Difficulties

(1) The reciprocal condition is not satisfied (2) treatment of the dilogarithmic function

Hydrodynamic Green's function

$$\boxed{G_H(\zeta,\zeta_0) = G(\zeta,\zeta_0) + S(\zeta_0)}$$

$$\varsigma(\eta) = -\frac{1}{4\pi^2\alpha}\int_0^\theta \frac{\alpha u - \sin u}{\alpha - \cos u}du - \frac{1}{4\pi^2\alpha}\log(-c)$$

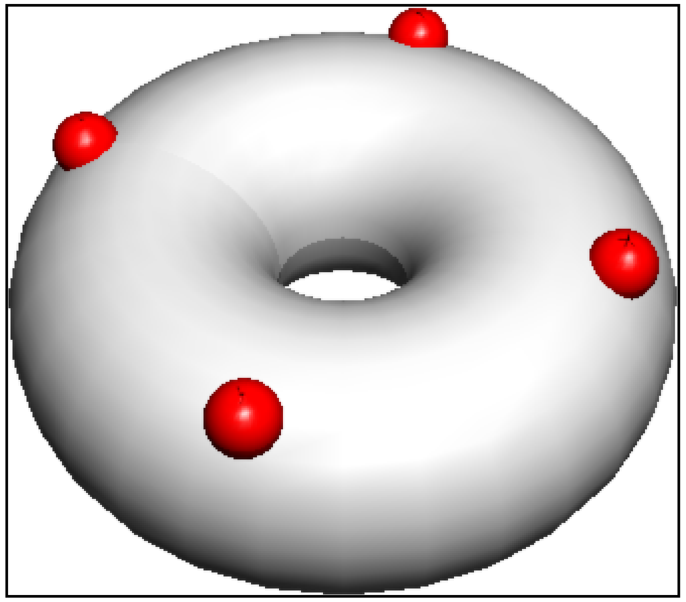
$$c = -\alpha - \mathcal{A}^{-1}$$

$$S(\theta) = -\frac{1}{4\pi^2\alpha}\int_0^\theta \frac{\alpha(u+\pi) - \sin u}{\alpha - \cos u}du$$

Evolution equations for point vortices

Robin function

$$R(\zeta_m) = \frac{1}{2\pi}\log\prod_{n\geq 1}(1-\rho^n)^2 + \varsigma(\eta_m) + \left(\frac{\log|\zeta_m|}{4\pi^2\mathcal{A}} - \frac{1}{4\pi}\right)\log|\zeta_m|$$
$$+S(\zeta_m) - \frac{1}{2\pi}\log\lambda(\zeta_m)|\zeta_m|$$



Location of N point vortices

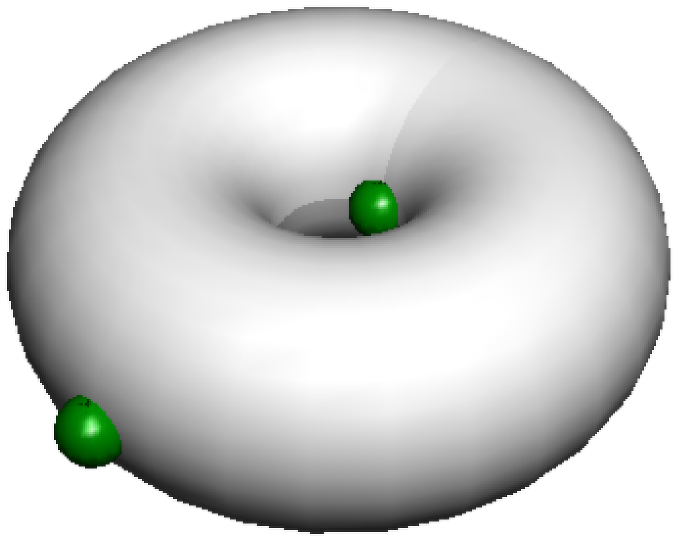
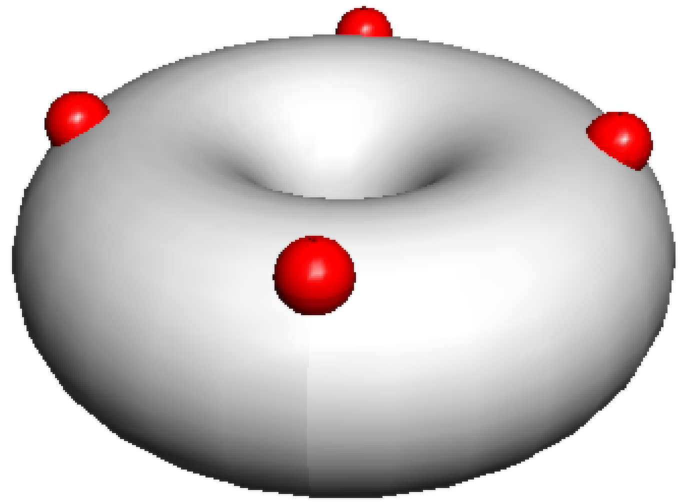
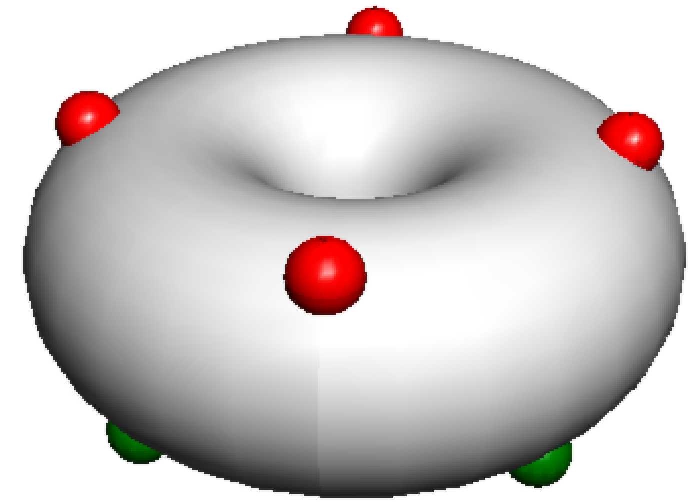
$$(\theta_m, \phi_m), m = 1, \ldots, N$$

Evolution equations for N point vortices

$$r^2(\alpha - \cos\theta_m)\frac{\mathrm{d}\theta_m}{\mathrm{d}t} = i\sum_{j\neq m}^{N}\Gamma_j\left[\frac{K(\zeta_m/\zeta_j) - \overline{K(\zeta_m/\zeta_j)}}{4\pi}\right],$$

$$r^2(\alpha - \cos\theta_m)^2\frac{\mathrm{d}\phi_m}{\mathrm{d}t} = \sum_{j\neq m}^{N}\Gamma_j\left[\frac{K(\zeta_m/\zeta_j) + \overline{K(\zeta_m/\zeta_j)}}{4\pi} + \frac{\alpha\theta_m - \sin\theta_m}{4\pi^2\alpha} + \frac{r_c(\theta_j)}{4\pi^2\mathcal{A}} - \frac{1}{4\pi}\right]$$
$$+\Gamma_m\left[\frac{\alpha\theta_m - \sin\theta_m}{4\pi^2\alpha} + \frac{r_c(\theta_m)}{4\pi^2\mathcal{A}} + \frac{1}{4\pi}\sin\theta_m\right].$$

Steady states

Antipodal locations	N-ring configuration	Pairs of M-ring
		
$(\theta,\phi)=(0,0)\quad(0,\pi)$ $(\pi,\pi)\quad(\pi,0)$	$(\theta_m,\phi_m)=(\Theta_0,\frac{2\pi m}{N})$	$(\theta_{2m-1},\phi_{2m-1})=(\Theta_1,\frac{2\pi m}{M})$ $(\theta_{2m},\phi_{2m})=(\Theta_2,\frac{2\pi m}{M}+\frac{\pi}{M})$ $(\theta_{2m-1},\phi_{2m-1})=(\Theta_1,\frac{2\pi m}{M})$ $(\theta_{2m},\phi_{2m})=(\Theta_2,\frac{2\pi m}{M})$ $\Theta_1+\Theta_2=2\pi$
Any strength	$\Gamma_m=\Gamma$	$\Gamma_{2m}=\Gamma\quad\Gamma_{2m-1}=-\Gamma$
Fixed equilibrium	Relative equilibrium	Relative equilibrium

Hamiltonian formulation

N-turple $M_N := M \times \cdots \times M$

Collision set $\mathcal{C} := \{q = (q_1, \cdots, q_N) \in M_N \mid \exists k \neq l, q_k = q_l\}$

Phase space

$$Q_N := M_N \setminus \mathcal{C}$$

2-form

$$\omega_N = \sum_{m=1}^{N} \Gamma_m \lambda^2 \circ p_m \, dx_{m,1} \wedge dx_{m,2}.$$

isothermal chart $(x_{m,1}, x_{m,2})$

Theorem. Let a function $\mathcal{H}$ on Q_N be define by

$$\mathcal{H}(q_1, \ldots, q_N) = -\frac{1}{2} \sum_{m=1}^{N} \sum_{j \neq m}^{N} \Gamma_m \Gamma_j G_H(\zeta_m, \zeta_j) - \frac{1}{2} \sum_{m=1}^{N} \Gamma_m^2 R(\zeta_m), \qquad (1)$$

then $(Q_N, \omega_N, \mathcal{H})$ is a Hamiltonian system. Moreover, the coordinate representation of $X_{\mathcal{H}}$ with respect to the isothermal chart $\{U, (x_{m,1}, x_{m,2})\}$ is given by

$$X_{\mathcal{H}} = \sum_{m=1}^{N} -\lambda^{-2}(x_{m,1}, x_{m,2}) \frac{\partial \psi_m}{\partial x_{m,2}} \frac{\partial}{\partial x_{m,1}} + \lambda^{-2}(x_{m,1}, x_{m,2}) \frac{\partial \psi_m}{\partial x_{m,1}} \frac{\partial}{\partial x_{m,2}}.$$

Integrability of 2-vortex problem

Hamiltonian is an invariant quantity

Corollary. As long as the solution exists, the Hamiltonian function $\mathcal{H}$ is invariant in time, namely,

$$\mathcal{H}(\theta_1(t), \phi_1(t), \ldots, \theta_N(t), \phi_N(t)) = \mathcal{H}(\theta_1(0), \phi_1(0), \ldots, \theta_N(0), \phi_N(0)).$$

The first integral

$$\boxed{I_N = \sum_{m=1}^{N} \Gamma_m(\alpha\theta_m - \sin\theta_m)}$$

Proposition. I_N is invariant in time and $\{I_N, \mathcal{H}\} = 0$.

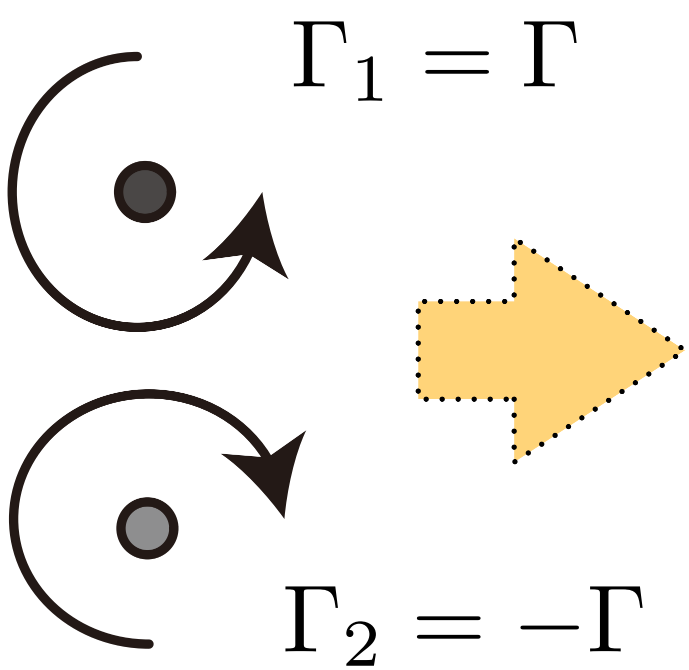
Integrability for N=2

Theorem. The 2-vortex problem is integrable for any vortex strengths.

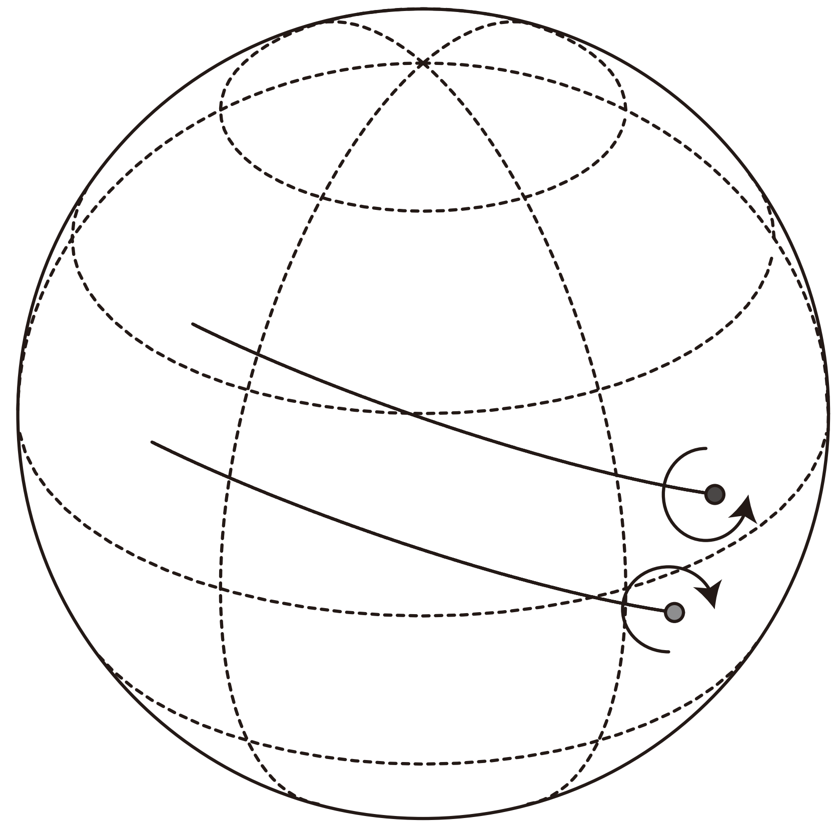
Motion of a vortex dipole

Vortex Dipole

On a sphere, it moves along the geodesic line



Locally it moves together



<u>What happens when it is in the toroidal surface?</u>

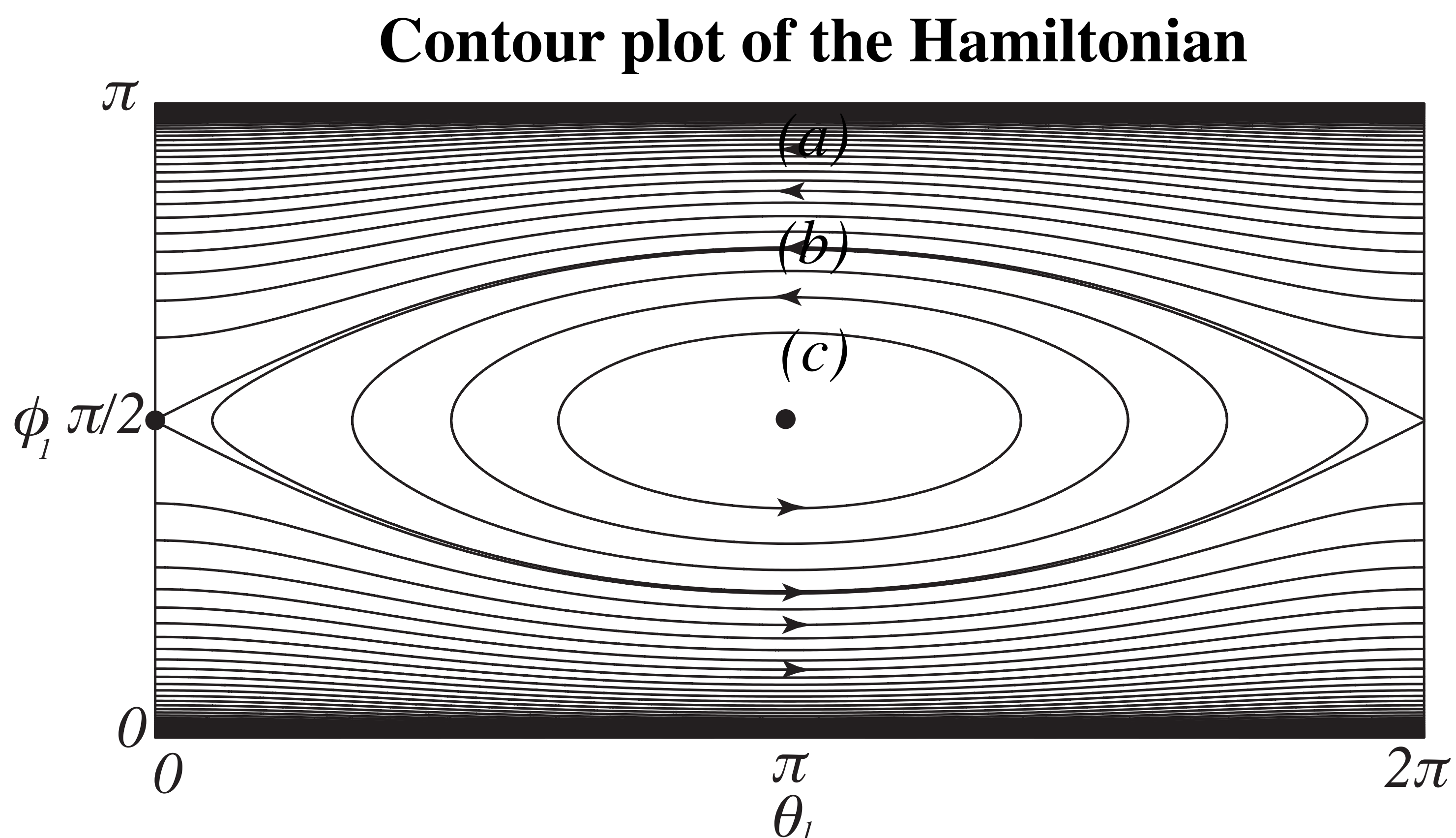
Theorem. The vortex dipole on the toroidal surface $\mathbb{T}_{R,r}$ never collides.

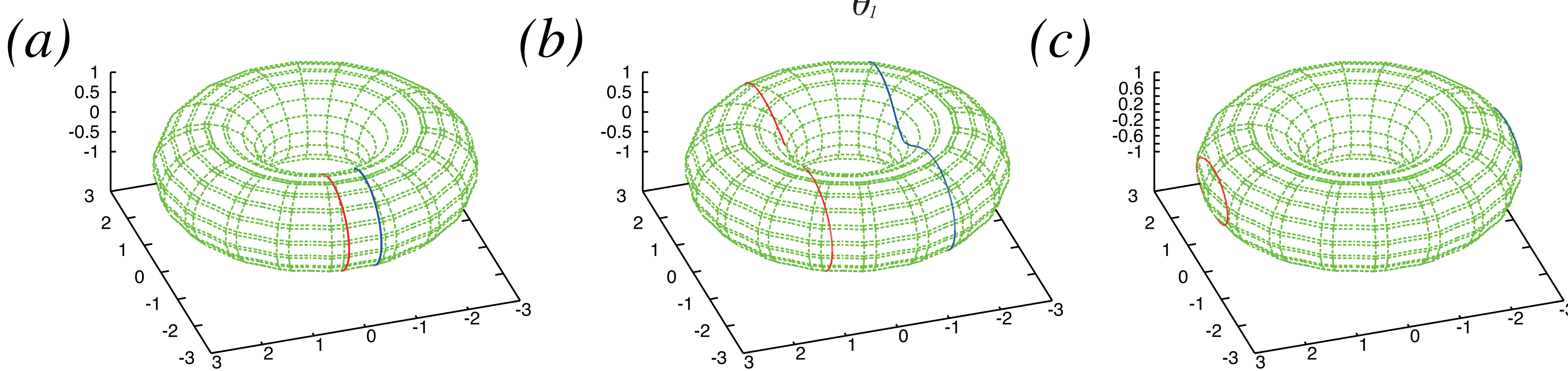
Lemma. $\forall t \in \mathbb{R}, \theta_1(t) = \theta_2(t),\ \phi_1(t) + \phi_2(t) = 0$ holds iff $\exists t_0 \in \mathbb{R}, \theta_1(t_0) = \theta_2(t_0)$ and $\phi_1(t_0) + \phi_2(t_0) = 0$.

Hamiltonian dynamics

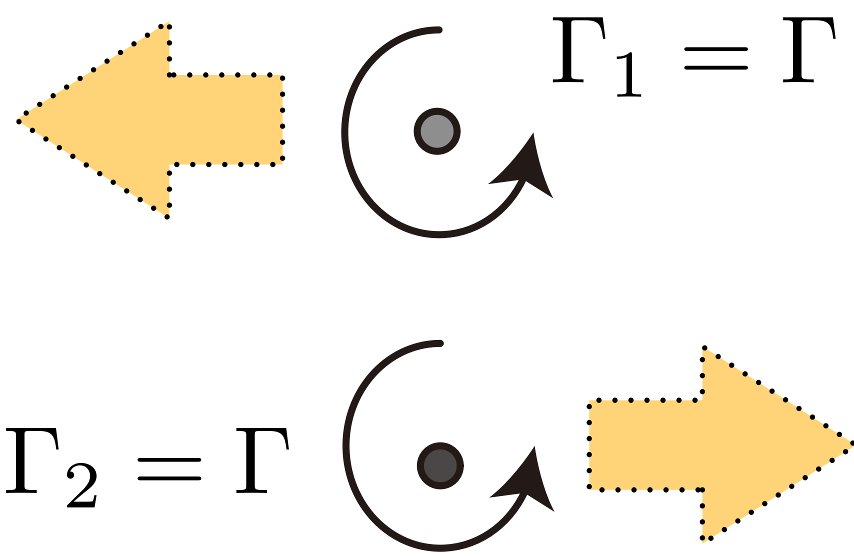
$$\forall t, \theta_1(t) = \theta_2(t),\ \phi_1(t) + \phi_2(t) = 0$$

$$\Longrightarrow \quad \boxed{\mathcal{H}(\theta_1, \phi_1) = \frac{\Gamma^2}{2\pi}\left[\log\left|P\left(e^{2i\phi_1}\right)\right| + \frac{1}{2}\log(R - r\cos\theta_1)\right]}$$

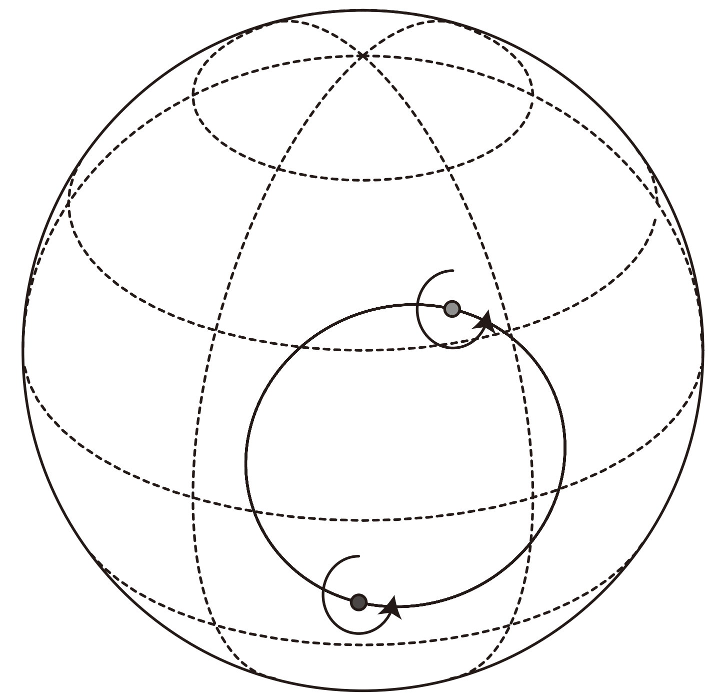




Identical vortex pair



On a sphere, the pair co-rotates



The first integral

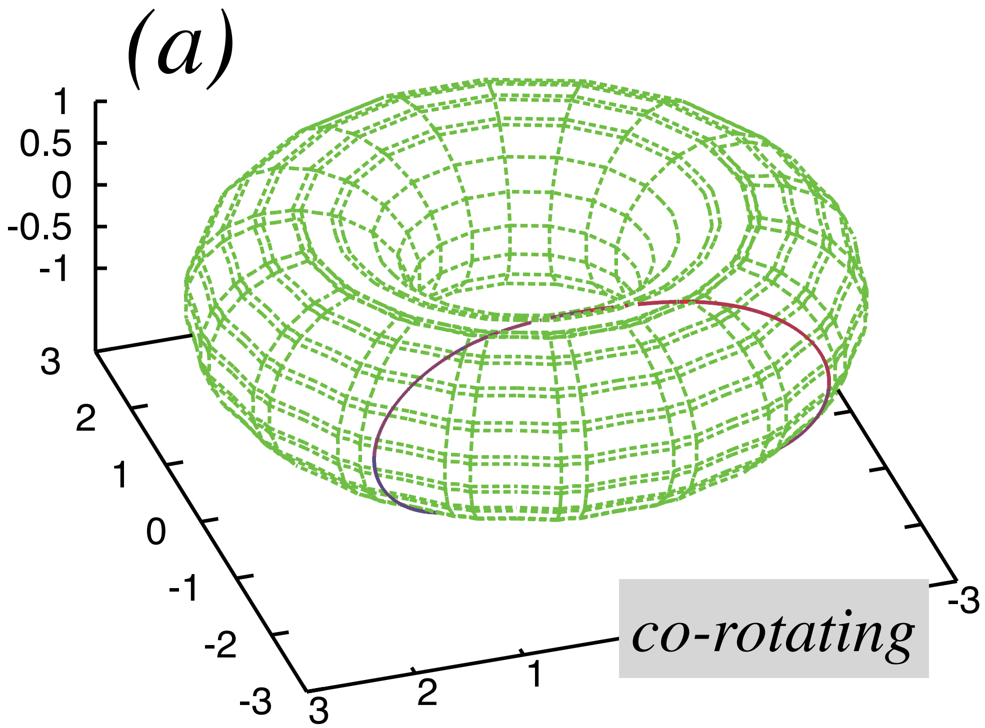
$$I_N = \sum_{m=1}^{2} \Gamma(\alpha\theta_m - \sin\theta_m)$$

Hamiltonian

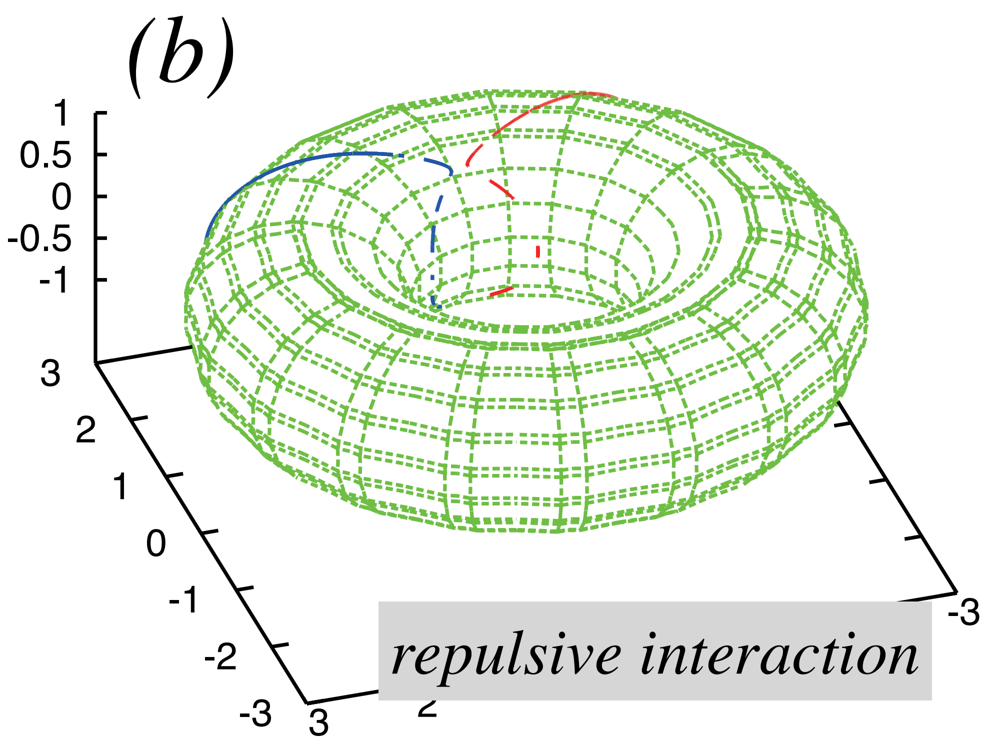
$$\mathcal{H}(\zeta_1, \zeta_2) = \boxed{-\frac{1}{2}\Gamma^2 G_H(\zeta_1, \zeta_2)} - \boxed{\frac{1}{2}\Gamma^2 R(\zeta_1)} - \boxed{\frac{1}{2}\Gamma^2 R(\zeta_2)}$$

function of $\Phi = \phi_1/\phi_2 \ \ \theta_1$ function of θ_1 function of θ_2

Motion of an identical vortex pair

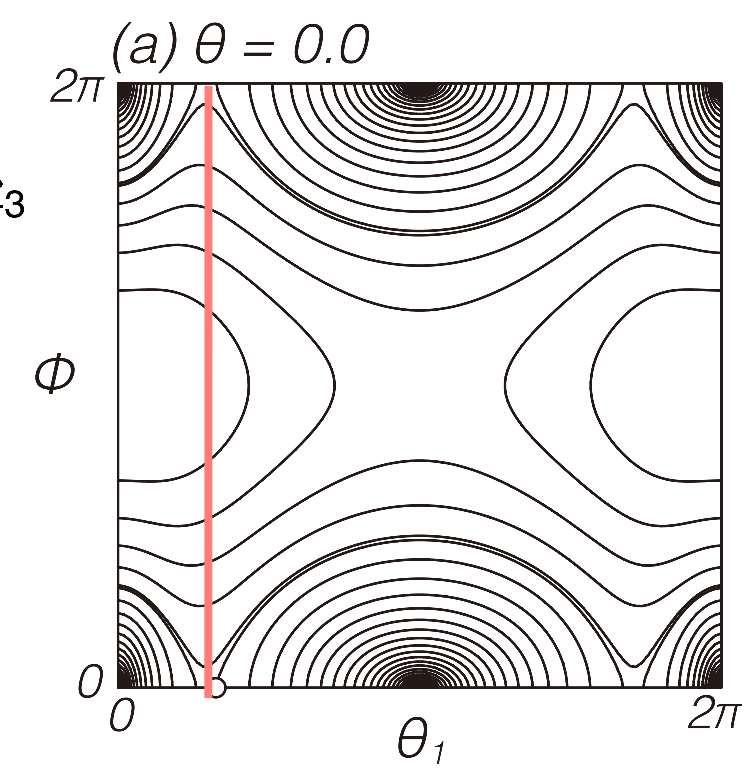


$$\Phi(0) = \phi_1(0) - \phi_2(0) = 0$$

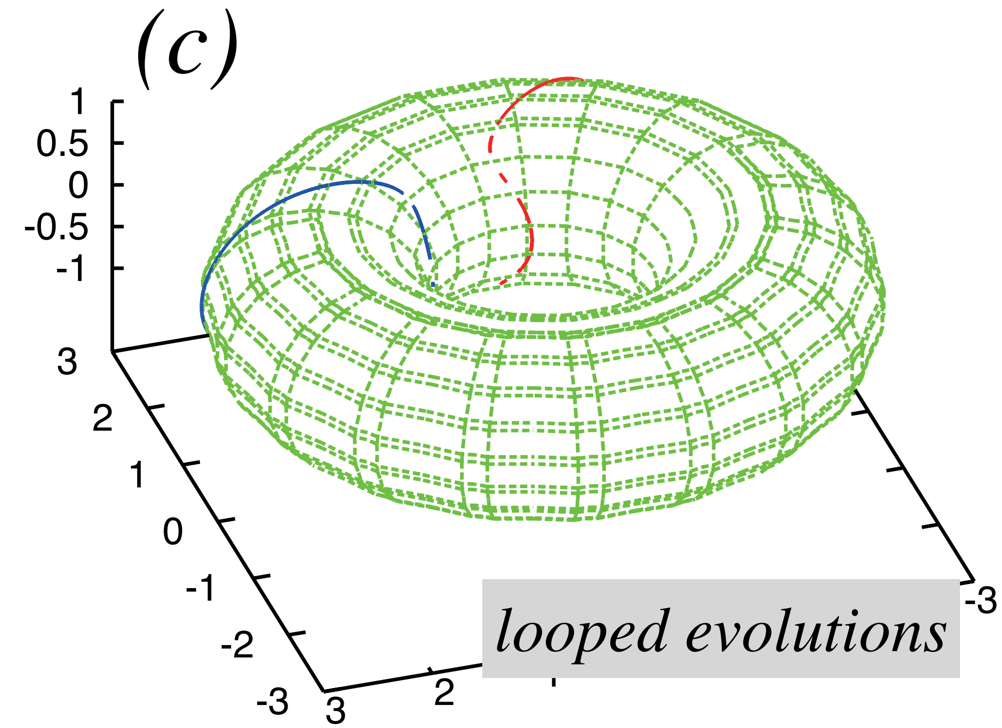


$$\Phi(0) = \phi_1(0) - \phi_2(0) = 0.075\pi$$

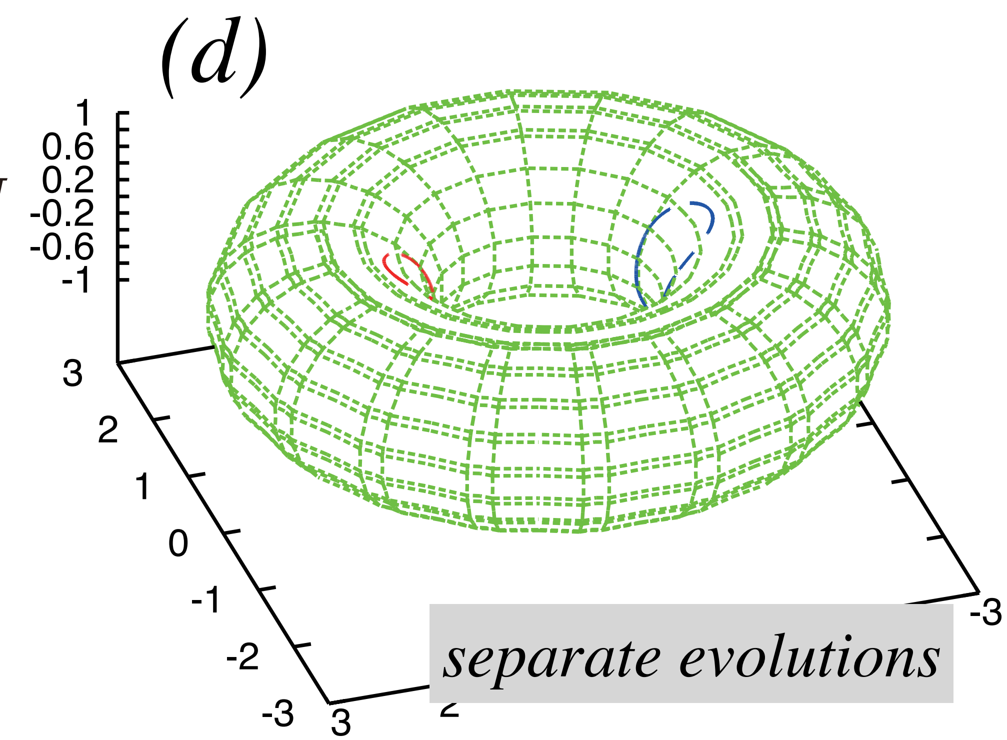
Contour plot



$$\theta_1(0) = \pi/3$$
$$\theta_2(0) = 2\pi - \pi/3$$



$$\Phi(0) = \phi_1(0) - \phi_2(0) = 0.25\pi$$



$$\Phi(0) = \phi_1(0) - \phi_2(0) = \pi$$

Summary

[1] The evolution equation of point vortices on a toroidal surface is derived.

[2] It is a Hamiltonian dynamical system. The 2-vortex problem is integrable.

[3] Orbit of a vortex dipole is going along an irreducible curve around the handle. Topological change of orbits is observed.

[4] Motion of an identical vortex pair can be repulsive, which is an unique orbit brought by the geometric nature of the torus.

[5] The stability of polygonal ring configuration of N point vortices (N-ring) is now in progress (and to be reported soon...)