Shape derivative of the contour integral type and its application to vortex patch equilibria

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Introduction: a vortex patch

Vortex patch = solution to Euler equation with uniform vorticity

Stationary 2D Euler equation

Flow domain $\Omega \subseteq \mathbf{C}$, vorticity $\omega \in \mathbf{R}$ and the level $\psi_0 \in \mathbf{R}$ of the unknown boundary

$$-\triangle \psi = \begin{cases} \omega & \text{in } D, \\ 0 & \text{in } \Omega \setminus \mathrm{cl}\, D, \end{cases}$$

$$\partial D = \{z \mid \psi = \psi_0\} = \text{streamline.}$$

$\Longrightarrow$ elliptic free-boundary problem

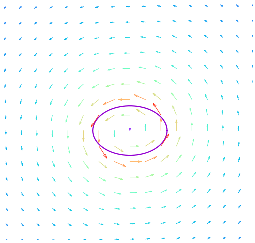
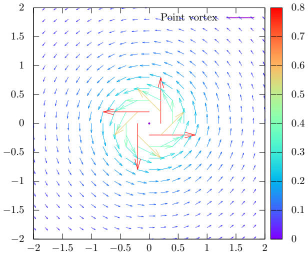


Figure: Velocity (u, v), streamline $\neq \partial D$

$$\omega \chi_D \xrightarrow[G\text{: Green's function}]{G * (\,\cdot\,)} \psi \xmapsto{2i\frac{\partial}{\partial z}} u - iv$$

Vortex patch

Vortex patch



A vortex patch induces:

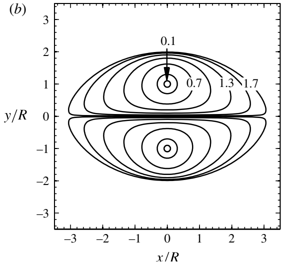
$$u - iv = \frac{\omega}{2\pi i} \iint_D \frac{1}{z - w} \, \mathrm{d}w_1 \, \mathrm{d}w_2.$$

A point vortex induces:

$$u - iv = \frac{\gamma}{2\pi i} \frac{1}{z - a}.$$

Preceding study: steadily translating vortex pair

Vortex patches D of vorticity $+\omega$ & $\overline{D}$ of $-\omega$



Reconstructed result of Pierrehumbert family [6]

Pierrehumbert's relaxation iter.

unknown boundary

∂D: $x = \pm g\,(y)$

equation

$\psi[g] = \psi_0$ on ∂D

(i.e. ∂D = streamline)

numerical scheme

Newton iteration on g

$+$ parameter-tuning iteration

Frèchet derivative of $\psi[g]$ w.r.t. g

Preceding study: Elcrat's trapped vortex

A vortex patch behind the cylinder in a uniform flow

Elcrat's method

unknown boundary

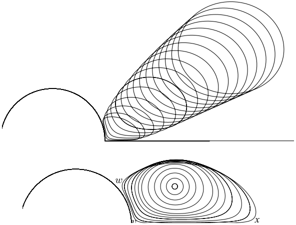
$$\partial D\colon\ z = z_0(s) + r(s)\, n_0(s)$$

equation

$$\underbrace{\mathrm{Re}[(u - iv)n]}_{\text{normal velocity}} = 0 \text{ on } \partial D$$

numerical scheme

Newton method



Numerical stability analysis: Elcrat, '05 [5]
↓ generalisation by shape calculus
Stability analysis: Elcrat&Protas, '13 [4]

The Jacobian matrix DF is used both for Newton method and for stability analysis.

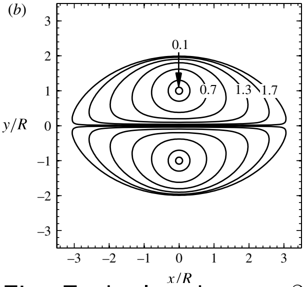
Steady vortex patch problem

Steady vortex patch problem

Find a vortex patch D whose shape does not change in time:

$$\mathcal{F}(\partial D) = 0.$$

It is a free-shape problem with unknown D.



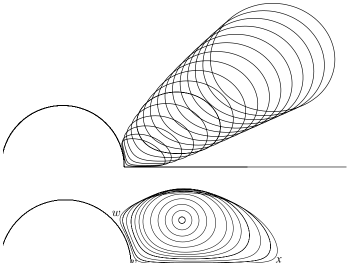


Fig: Each closed curve ∂D corresponds to a steady solution.

Goal

Preceding study: Stability analysis framework by Elcrat&Protas

- They consider only planar flows.
- Singularities are removed, whereafter the shape derivative formula of the boundary integral type is applied.

$$-\oint_C \log\left(z-w\right) \mathrm{d}\overline{w} = \oint_C \frac{\overline{z}-\overline{w}}{z-w} \mathrm{d}w.$$

$\Longrightarrow$ too complicated

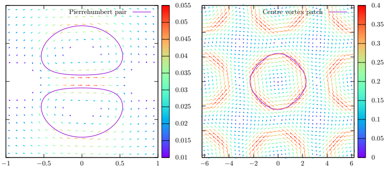
My works

- Shape derivative of the contour integral type
- Shape derivative of the singular contour integral type
- $\Longrightarrow$ applicable to an arbitrary flow domain

ToC

Steady vortex patch problem

Find a vortex patch D whose shape does not change in time:
$$\mathcal{F}(\partial D) = 0.$$



Section 2

Shape derivative

Subsections

Why shape derivative?

$$\mathcal{F}\left(\partial D\right) = 0$$

$$u - iv = \frac{\omega}{2\pi i} \iint_{D} \frac{1}{z - w} \, \mathrm{d}w_1 \, \mathrm{d}w_2 = \frac{-\omega}{4\pi} \oint_{\partial D} \log\left(z - w\right) \mathrm{d}\overline{w}.$$

Idea

Newton's method & shape derivative of $\mathcal{F}$ [5, 4]

Shape derivative in terms of Gâteaux derivative

Definition. Gâteaux (semi)derivative

$$\begin{aligned} \mathrm{d}\mathcal{F}\left(f;\delta f\right) &:= \lim_{h\to+0}\frac{1}{h}\left(\mathcal{F}\left(f+h\delta f\right)-\mathcal{F}\left(f\right)\right) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}h}\right|_{h=0}\mathcal{F}\left(f+h\delta f\right). \end{aligned}$$

$$\begin{aligned} J\left(D\right) &:= \int_D \varphi\left(x\right)\mathrm{d}\mu\left(x\right) \\ &\implies \quad \mathrm{d}J\left(D;V\right) = \int_D \nabla\cdot\left(\varphi\left(x\right)V\left(x\right)\right)\mathrm{d}\mu\left(x\right), \\ J\left(\partial D\right) &:= \int_{\partial D}\varphi\left(x\right)\mathrm{d}x \\ &\implies \quad \mathrm{d}J\left(D;V\right) = \int_{\partial D}\nabla\varphi\cdot V+\varphi(\nabla\cdot V-J_xVn\cdot n)\,\mathrm{d}x. \end{aligned}$$

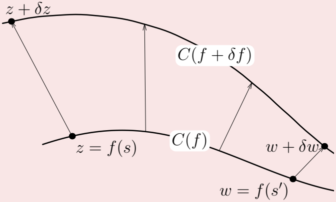
These are not applicable to our cases.

Shape derivative of contour integral

Theorem. Shape derivative of contour integral

Let $\varphi(z, \overline{z})$ be a C^1 function.
We denote by $C = C(f)$ a curve with parametrisation $z = f(s)$, $f \in C^1([0,1]; \mathbf{C})$. We define

$$\mathcal{F}(f) := \int_{C(f)} \varphi(z, \overline{z})\, \mathrm{d}z.$$



Then the Gâteaux semiderivative at f in the direction δf is

$$\mathrm{d}\mathcal{F}(f; \delta f) = \varphi\left(b, \overline{b}\right) \delta b - \varphi(a, \overline{a})\, \delta a + 2i \int_{C(f)} \frac{\partial \varphi}{\partial \overline{z}} \mathrm{Re}[i \delta z\, \mathrm{d}\overline{z}].$$

where $\delta z = \delta f(s)$, $a = f(0)$, $b = f(1)$, $\delta a = \delta f(0)$, $\delta b = \delta f(1)$.

Shape derivative of contour integral

$$\mathrm{d}\mathcal{F}\left(f;\delta f\right) = \varphi\left(b,\overline{b}\right)\delta b - \varphi\left(a,\overline{a}\right)\delta a + 2i\int_{C(f)} \frac{\partial\varphi}{\partial\overline{z}}\,\mathrm{Re}[i\delta z\,\mathrm{d}\overline{z}].$$

E.g. $\int$ of holomorphic fun.

depends only on endpoints
($\because \varphi_{\overline{z}} = 0$)

E.g. expr. of $\int$ by parametrisation

does not depend on parametrisations
($\because \delta z \perp -if'$)

E.g. the shape derivative of the area

$$\mathcal{A}\left(f\right) := (\text{the area surrounded by } C\left(f\right)) = \frac{1}{2i}\oint_{C(f)} \overline{z}\,\mathrm{d}z$$

$$\overset{\text{S.D.}}{\Longrightarrow}\ \mathrm{d}\mathcal{A}\left(f;\delta f\right) = \mathrm{Re}\langle\delta f, -if'\rangle_{L^2}.$$

Shape derivative of singular contour integral

Theorem. Shape derivative of singular contour integral

Under certain assumptions on logarithmic singularities of $\varphi\left(z,\overline{z},w,\overline{w}\right)$ at $z=w$, we have:

$$\mathcal{F}\left(f\right) := \oint_{C(f)} \varphi\left(z,\overline{z},w,\overline{w}\right) \mathrm{d}w$$

$$\implies \quad \mathrm{d}\mathcal{F}\left(f;\delta f\right) = 2i \oint_{C(f)} \frac{\partial\varphi}{\partial\overline{w}} \operatorname{Re}[i(\delta w-\delta z)\,\mathrm{d}\overline{w}],$$

where $\delta z := \delta f\circ f^{-1}\left(z\right)$ and $\delta w := \delta f\circ f^{-1}\left(w\right)$.

Shape derivative of singular contour integral

Shape derivative of contour integral with $\log$ kernel

$$-\frac{\omega}{4\pi}\mathrm{p.v.}\oint_C \log\left(z-w\right)\mathrm{d}\overline{w}$$

$$\xRightarrow{\text{S.D.}} \quad -\frac{\omega}{2\pi i}\oint_C \frac{\mathrm{Re}[i(\delta z-\delta w)\,\mathrm{d}\overline{w}]}{z-w}.$$

The obtained integrand admits of the continuous modification. We can thus compute this integral with high accuracy.

Section 3

Steady vortex patch

Subsections

Numerical method for vortex patch equilibria

unknown boundary

$$z = z_0(s) + r(s)\, n_0(s) =: Z(r)(s)$$

equation

$$\mathcal{F}(f) := \mathrm{Re}[(u - iv)(-if')] = 0 \text{ on } C(f)$$

(normal velocity vanishes on $C(f)$)

$\Longrightarrow$ Newton method

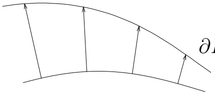


Figure: $z = z_0 + rn_0$

The Jacobian matrix is obtained by S.D. formula and chain rule:

$$\mathcal{F} \circ Z(r) = \mathrm{Re}\Big[(u - iv)|_{z=Z(r)} \left(-iZ(r)'\right) \Big]$$

$$\stackrel{\text{S.D.}}{\Longrightarrow} \quad \mathrm{d}(\mathcal{F} \circ Z)(r; \delta r) \stackrel{\text{chain rule}}{=} \underbrace{\mathrm{d}\mathcal{F}}_{\text{Thm}}(Z(r); \delta r n_0).$$

Some reconstructed numerical results

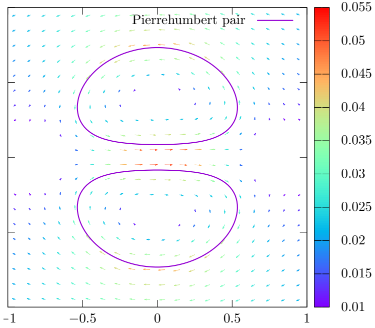


Figure: Translating pair [6]

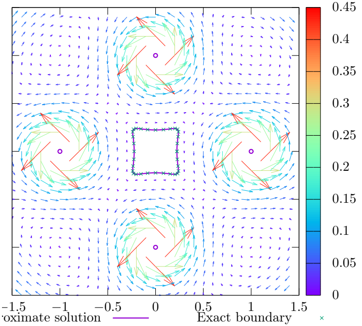


Figure: Crowdy's exact solution [1]

Vortex lattices

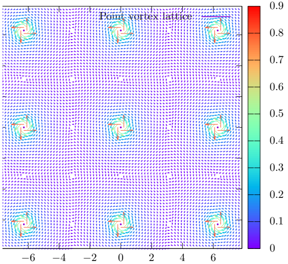


Figure: point vortex lattices

[7, Tkachenko, '66]

- use of Weierstrass ζ fun.
- rigidly rotating lattices
- doubly-periodic in frame of rot.

[2, Crowdy, '10]

- use of Schottky-Prime fun. P
- rapidly convergent Laurent series

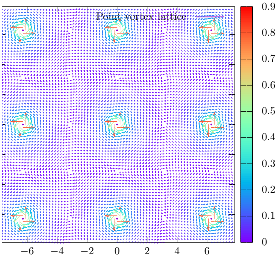
$\Longrightarrow$ How about vortex patch lattices?

Point vortex lattices

a complex potential

$$\log P(\zeta) \quad \text{under} \quad z = -i \log \zeta.$$

P function [2] has zeros at $z = z_{km}$.
K function, the $\log$ derivative of P, has singularities at z_{km}.



Assumption. induced velocity field

$$u - iv = \frac{\Gamma}{2\pi} K(\zeta) + \alpha z + \beta.$$

Assumption. rigidly rotation

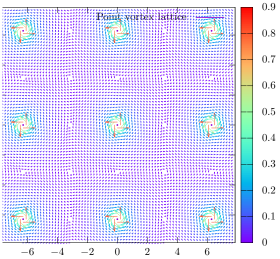
$$(u + iv)\big|_{z=z_{km}} = i\Omega z_{km}$$

$\xRightarrow{\text{determine}}$ constants $\alpha, \beta \in \mathbf{C}$.

Point vortex lattices

a complex potential

$$\log P\left(\zeta\right) \quad \text{under} \quad z = -i \log \zeta.$$



$$\Omega = \frac{\Gamma}{2 \times (\text{area of period window})},$$
$$\alpha = -i\Omega,$$
$$\beta = -\frac{\Gamma}{4\pi}.$$

$u - iv = \dfrac{\Gamma}{2\pi}\left[K\left(\zeta\right) - \dfrac{1}{2}\right] - i\Omega z$ is doubly periodic in the co-rotating frame of the angular velocity Ω.

Vortex patch lattices

$$\log P(\zeta) \quad \text{under} \quad z = -i \log \zeta,$$

$$u - iv = \frac{\omega}{2\pi} \iint_D \left[K\left(\exp\left(iz - iw\right)\right) - \frac{1}{2} \right] \mathrm{d}w_1 \, \mathrm{d}w_2 - i\Omega z,$$

$$\Gamma = \omega |D| = 2\Omega \times \text{(area of period window)}.$$

- $u - iv$ has a doubly periodicity
- Green's formula yields logarithmic singular integral
  $\Longrightarrow$ the shape derivative formula is applicable.

Regular lattice arrangement

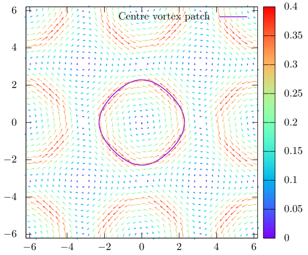


Figure: Velocity field induced by the vortex patch lattices

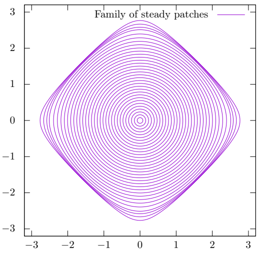
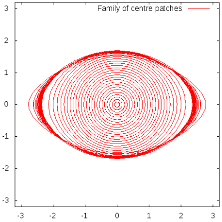
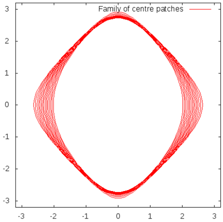


Figure: Numerical solutions ∂D for varying ω

Non-regular lattice arrangement (aspect ratio < 1)





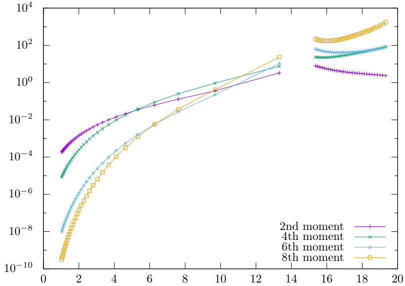
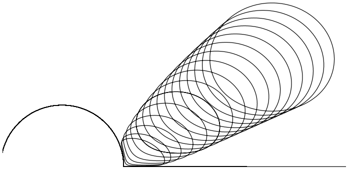
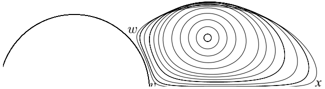


Figure: $2k$-th moments of the shapes are plotted against the areas.

Summary

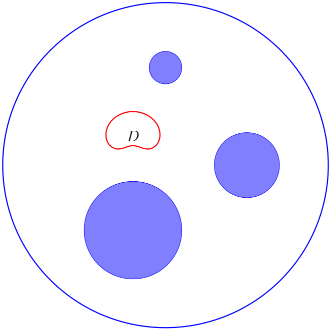
- simpler shape derivative formula of the contour integral type
- numerical results for vortex patch equilibria
- applicable to $\log$ singularities
  - e.g. doubly periodic case
  - task: an arbitrary flow domain
  - task: stability analysis as in [3, 4]





Future works

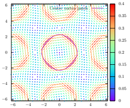
$$W\left(z\right)=\frac{1}{2\pi i}\iint_{D}\log\underbrace{\omega\left(z,w\right)}_{\text{S-K prime}}\mathrm{d}w_1\,\mathrm{d}w_2+\dots$$

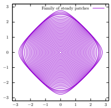


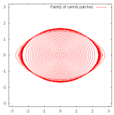
Thank you for kind attensions!

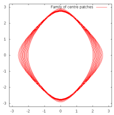
$$\mathcal{F}(f) := \oint_{C(f)} \varphi(z, \overline{z}, w, \overline{w})\,\mathrm{d}w$$

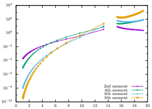
$$\Longrightarrow \quad \mathrm{d}\mathcal{F}(f; \delta f) = 2i \oint_{C(f)} \frac{\partial \varphi}{\partial \overline{w}} \mathrm{Re}[i(\delta w - \delta z)\,\mathrm{d}\overline{w}].$$











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Shape derivative of singular contour integral

Let D be a simply connected domain in $\mathbf{C}$ and $\varphi(z,w)$ be a map from $D \times D \setminus \{z \neq w\}$ to $\mathbf{C}$. Assume, for z and w along an arbitrary curve C in D, $\varphi(z,w)|z-w|$, $\varphi_w(z,w)|z-w|$ and $\varphi_{\overline{w}}(z,w)|z-w|$ have continuous modifications as $z \to w$. For a parametrisation f with $C(f) \subset D$, we define the map $\mathcal{F}$ of f as follows:

$$\mathcal{F}(f) := \oint_{C(f)} \varphi(z,\overline{z},w,\overline{w})\,\mathrm{d}w\Big|_{z=f(\,\cdot\,)}.$$

We then have the following:

$$\begin{aligned}\mathrm{d}\mathcal{F}(f;\delta f) = &\oint_C \mathrm{d}\varphi(z,w;\delta z,\delta z)\,\mathrm{d}w \\ &+ 2i\oint_C \varphi_{\overline{w}}\,\mathrm{Re}\Big[(\delta w-\delta z)\overline{(-i\,\mathrm{d}w)}\Big],\end{aligned}$$

where $\delta z := \delta f \circ f^{-1}(z)$ and $\delta w := \delta f \circ f^{-1}(w)$.

Lagrangian approach v.s. geometric approach

The Lagrangian approach in which one follows the trajectory of a fluid particle on the vortex boundary; we note that displacement of a material particle in general also involves a component tangential to the vortex boundary, which does not lead to deformations of the contour; and the geometric approach, in which one considers displacements of the points on the boundary in the direction normal to the boundary only.

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