Double Field Theory

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Contents

1 Introduction

String theory proposes that the elementary particles are string-like one dimensional objects instead of point particles as in the Standard Model [1]. A distinct characteristic of such an object is that a string can wrap around in non-contractible cycles. This 'wrapping' introduces winding state which has no counterpart in point particle field theories. The presence of winding state together with momentum leads to a T-duality symmetry, representing a physical equivalence of string theories of different formulations [2].

For example, for closed strings there is a T-duality between the two type II theories. The type IIA theory with small radius $R_A \to 0$ is physically equivalent to the type IIB theory with a large radius $R_B \to \infty$, and vice-versa [T-intro p.8]. This equivalence is commonly referred to as 'the physics at a very small scale cannot be distinguished from the physics at a very large scale'. It is also interesting that the two heterotic string theories $SO(32)$ and $E_8 \times E_8$ are also related via T-duality [1]. For simplicity, it should be noted that only the massless sector will be considered in this dissertation.

One of the methods to incorporate T-duality naturally into string theory is double field theory which arises from toroidal compactification of string theory [4,7,8]. Toroidal compactification of a D-dimensional theory to d non-compact and n compact dimensions such that $D = d + n$ results in a D-dimensional momentum and an n-dimensional winding modes. Then we can assign coordinates dual to the momentum, $x^i = (x^\mu, x^a)$ where $i = 1, ..., D,\ \mu = 1, ..., d,\ a = 1, ..., n$. Double field theory proposes that there are dual coordinates to the winding states as well, namely $\tilde{x}^a$. Such addition of coordinates leads to a 'doubled torus' geometry which could enhance the understanding of the physics beyond conventional string theory.

This dissertation consists of a brief introduction of T-duality, its formulation in toroidal compactification, the basics of double field theory and the nature of how T-duality is incorporated in double field theory. Finally, the dimensional reduction of double field theory will be discussed.

2 T-duality

In this section the basic concept of T-duality in bosonic closed string theory and its form in toroidal compactification are explored.

2.1 One-loop Closed Bosonic String

Consider a closed string in a 26-dimensional Minkowski spacetime with the one direction compactified in a circle of radius R, such that

$$X^i(\tau,\sigma) = X^i(\tau,\sigma+\pi), \tag{2.1.1}$$

with period identification

$$X^{25} \approx X^{25} + 2\pi R m, \tag{2.1.2}$$

where $i = 1, ..., 26$, and m is an arbitrary integer. Since the number of wraps around the circle depends on m, it is called the winding number. Winding number $m = 1$ corresponds to a $2\pi R$ shift in the X^{25} direction.

The closed string mode expansion with taking care of the periodic condition has the form

$$X^{25}(\tau,\sigma) = x^{25} + \alpha' p^{25}\tau + Rm\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{1}{k}(\alpha_k^{25}e^{ik\sigma} + \tilde{\alpha}_k^{25}e^{-ik\sigma})e^{-ik\tau}, \tag{2.1.3}$$

where α' is the Regge slope and $\alpha_k, \tilde{\alpha}_k$ are harmonic oscillators [1,2]. Hence the first three terms of the expansions represent the centre of mass motion of the string where as the summation term represents the oscillation of the string.

Now because x^{25} is compact, the corresponding momentum p^{25} should be quantised. This is due to the fact that as in quantum mechanics the wavefunction includes a factor e^{ipx} and to maintain the initial value, increasing x should be compensated by decreasing p. Hence the quantised momentum has the form

$$p^{25} = \frac{n}{R}, \tag{2.1.4}$$

where n is an arbitrary integer. Note the momentum in the uncompactified dimensions are still continuous.

A property of a closed string is that it holds left-moving and right-moving waves which correspond to holomorphic and antiholomorphic fields. Hence one can divide the mode expansion of $X^{25}(\tau,\sigma)$ into two parts

$$X^{25}(\tau,\sigma) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma), \tag{2.1.5}$$

which have the form

$$\begin{aligned} X_L^{25}(\tau+\sigma) &= x_L^{25} + \sqrt{\frac{\alpha'}{2}} p_L^{25}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k} \tilde{\alpha}_k^{25} e^{-ik(\tau+\sigma)} \\ X_R^{25}(\tau-\sigma) &= x_R^{25} + \sqrt{\frac{\alpha'}{2}} p_R^{25}(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{k\neq 0} \frac{1}{k} \alpha_k^{25} e^{-ik(\tau-\sigma)}, \end{aligned} \tag{2.1.6}$$

where $x = x_L + x_R$ and the momentum values are

$$\begin{aligned} p_L^{25} &= \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R}n + \frac{R}{\sqrt{\alpha'}}m) \\ p_R^{25} &= \frac{1}{\sqrt{2}}(\frac{\sqrt{\alpha'}}{R}n - \frac{R}{\sqrt{\alpha'}}m). \end{aligned} \tag{2.1.7}$$

Hence the left and right Viraroso operators [10] take the form

$$\begin{aligned} L_0 &= \frac{\alpha'}{4}((p_L^{25})^2 + p^2) + \sum_{k=1}^{\infty} \tilde{\alpha}_{-k}\tilde{\alpha}_k \\ \bar{L_0} &= \frac{\alpha'}{4}((p_R^{25})^2 + p^2) + \sum_{k=1}^{\infty} \alpha_{-k}\alpha_k. \end{aligned} \tag{2.1.8}$$

where $p^2 = \sum p^\mu p_\mu,\ \mu = 1, ..., 24$ and we define $\sum \tilde{\alpha}_{-k}\tilde{\alpha}_k = N, \quad \sum \alpha_{-k}\alpha_k = \bar{N}$. Note the normal ordered Hamiltonian is the sum of the two operators $H = L_0 + \bar{L_0}$ [2].

A state should satisfy two important constraints to be considered as physical in string theory. One is level matching condition

$$L_0 - \bar{L}_0 = 0, \tag{2.1.9}$$

and the other is the mass-shell constraint (for free string in this context)

$$L_0 = 1, \quad \bar{L}_0 = 1. \tag{2.1.10}$$

With the mass formula is given by

$$m^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu = -p^2, \tag{2.1.11}$$

the mass of the string can be deduced from (2.1.8) and (2.1.10) as

$$m^2 = \frac{\alpha'}{R^2}n^2 + \frac{R^2}{\alpha'}m^2 + \frac{2}{\alpha'}(N + \bar{N} - 2), \tag{2.1.12}$$

where the first two terms on the righthand side correspond to the contributions from the momentum, p_{25}, and winding mode, w, to the mass in the compactified dimenions. Note the momentum value of the compactified dimension is not included in the mass formula. Using the same method, the level matching condition can be rewritten as

$$N - \bar{N} = p_{25}w. \tag{2.1.13}$$

Now it is simple to see that L_{0L}, L_{0R} and m^2 are invariant under the transformation

$$\frac{R}{\sqrt{\alpha'}} \to \frac{\sqrt{\alpha'}}{R}, \qquad m \leftrightarrow n. \tag{2.1.14}$$

Under this transformation p_L and $\tilde{\alpha}_k$ are invariant whereas $p_R \to -p_R$ and $\tilde{\alpha}_k \to -\tilde{\alpha}_k$. The above is a simple example of a target space duality, where interchanging momenta and winding modes with inverting the radius gives an equal physics.

The Hamiltonian $H = L + \bar{L}$, hence the energy is also invariant under such symmetry. Energy contribution from winding mode can be understood this way - as a string is wound around a non-contractible circle of radius R, it needs to be stretched thus introducing additional potential energy as the winding number increases. This contribution can be calculated by the m^2R^2 term.

For compactification of more than one dimensions (2.1.12) and (2.1.13) can be rewritten as [3]

$$m^2 = \frac{\alpha'}{R^2}p^2 + \frac{R^2}{\alpha'}w^2 + \frac{2}{\alpha'}(N + \bar{N} - 2) \tag{2.1.15}$$

$$N - \bar{N} = p_a w^a \tag{2.1.16}$$

where p_a are momenta and w^a are winding modes, $a = 1, ..., n$ label the n compactified dimensions.

2.2 T-duality in Toroidal Compactifications

It is well investigated that T-duality can be generalised to $O(D, D, \mathbb{Z})$ group acting on the moduli space of toroidal compactifications [4]. Here the action of $O(D, D, \mathbb{Z})$ group under constant background metric and antisymmetric tensor is briefly discussed.

An element $h \in O(D, D, \mathbb{Z})$ is a $2D \times 2D$ matrix that preserves the $O(D, D, \mathbb{Z})$ invariant metric J such that $h^t J h = J$ where

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{2.2.1}$$

In a D-dimensional toroidal compactification, the momentum and winding modes are both D-dimensional objects p^i and w_i. They can be merged into a single object called the generalised momentum

$$Z^M = \begin{pmatrix} w_i \\ p^i \end{pmatrix}, \tag{2.2.2}$$

which is a $2D$-dimensional column vector. As it will be explained later, the Hamiltonian can be expressed as

$$H = \frac{1}{2} Z^t \mathcal{R}(E) Z + N + \bar{N}, \tag{2.2.3}$$

where $\mathcal{R}(E)$ is a $2D \times 2D$ metric [11,12]

$$\mathcal{R}(E) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.2.4}$$

The argument E is the background matrix defined as $E_{ij} = G_{ij} + B_{ij}$ where G_{ij} is the background metric and B_{ij} is the background antisymmetric tensor. Likewise, the mass formula and level matching condition can be expressed as

$$M^2 = Z^t \mathcal{R}(E) Z + (N + \bar{N} - 2), \tag{2.2.5}$$

$$\tfrac{1}{2} Z^t J Z = N - \bar{N}. \tag{2.2.6}$$

$O(D, D, \mathbb{Z})$ has three types of generators, i.e.) any element h can be decomposed into products of the generators [2,13,14,15].

- **Diffeomorphism**: It is possible to change the basis of the compactified dimensions thus conjugating the background matix $E' = AEA^t$ where A represents the basis change. The corresponding $O(D, D, \mathbb{Z})$ elements are of the form

$$h_A = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \tag{2.2.7}$$

Note A itself is an element of the group $GL(d,\mathbb{Z})$.

For example,

$$Z^t\mathcal{R}(E')Z = p^t(A^{-1})^t G^{-1}A^{-1}p - w^t A(G - BG^{-1}B)A^t w \\ - 2p^t(A^{-1})^t(G^{-1}B)A^t w \tag{2.2.8}$$

can easily be mapped back into $Z^t\mathcal{R}(E)Z$ by redefining $p' = A^{-1}p,\ w' = A^t w$.

- Θ **Shift**: One can shift the antisymmetric tensor B_{ij} with an antisymmetric matrix Θ_{ij} where $\Theta_{ij} \in (\mathbb{Z})$, i.e.) the entries in the matrix are integers. The corresponding $O(D,D,\mathbb{Z})$ elements are of the form

$$h_\Theta = \begin{pmatrix} I & \Theta \\ 0 & I \end{pmatrix}. \tag{2.2.9}$$

- **Factorised T-duality**: The factorised T-duality action is the generalisation of $R \to 1/R$ duality illustrated in section (2.1). Its matrix form is given as

$$h_T^{(k)} = \begin{pmatrix} I - t_k & t_k \\ t_k & I - t_k \end{pmatrix}. \tag{2.2.10}$$

where $t_k = diag(0...0\ 1\ 0...0)$ and the 1 represents the k-direction in which T-duality is acted. As long as the background metric and the antisymmetric tensor are constant, the factorised T-duality action in the k-direction triggers a set of transformations called Buscher rules [16,17]

$$G_{kk} \to \frac{1}{G_{kk}},\quad G_{ki} \to \frac{B_{ki}}{G_{kk}},\quad G_{ij} \to G_{ij} - \frac{G_{ki}G_{kj} - B_{ki}B_{kj}}{G_{kk}}, \\ B_{ki} \to \frac{G_{ki}}{G_{kk}},\quad B_{ij} \to B_{ij} - \frac{G_{ki}B_{kj} - B_{ki}G_{kj}}{G_{kk}}. \tag{2.2.11}$$

It is interesting to note that the metric in the k-direction - G_{kk} - is inverted, similar to the $R \to 1/R$ transformation.

A demonstration of how the theory is $O(D,D,\mathbb{Z})$-invariant may well involve the Hamiltonian. First consider the worldsheet action in a toroidal background

$$S = \frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \Big(\sqrt{g}g^{\alpha\beta}G_{ij}\partial_\alpha X^i\partial_\beta X^j + \epsilon^{\alpha\beta}B_{ij}\partial_\alpha X^i\partial_\beta X^j \\ -\frac{1}{2}\sqrt{g}\Phi R\Big), \tag{2.2.12}$$

where the toroidal compactification indicates $X^i \approx X^i + 2\pi w^i$. Hence the mode expansion of coordinates and conjugate momenta can be calculated as

$$X^i(\tau,\sigma) = x^i + w^i\sigma + \tau G^{ij}(p_j - B_{jk}w^k) + \frac{i}{\sqrt{2}}\sum_{l\neq 0}\frac{1}{l}[\alpha_l^i(E)e^{-il(\tau-\sigma)} + \tilde{\alpha}_l^i(E)e^{-il(\tau+\sigma)}] \tag{2.2.13}$$

$$2\pi P_i(\tau,\sigma) = p_i + \frac{1}{\sqrt{2}}\sum_{l\neq 0}[E_{ij}^t\alpha_l^j(E)e^{-il(\tau-\sigma)} + E_{ij}\tilde{\alpha}_l^j(E)e^{-il(\tau+\sigma)}] \tag{2.2.14}$$

The first term of the Hamiltonian (2.2.3) has already been proved to be invariant under $O(D,D,\mathbb{Z})$ in (2.2.8). Thus for completion the transformations of the number operators $N, \bar{N}$ of the Hamiltonian need to be addressed.

First the action of $O(D,D,\mathbb{Z})$ on the background E needs to be illustrated. Since $O(D,D,\mathbb{Z})$ is a $2D\times 2D$ matrix and E is a $D\times D$ matrix, they cannot be simply multiplied. To deal with this problem fractional linear transformation is used, where the action is defined as [4]

$$E' = h(E) \equiv (aE+b)(cE+d)^{-1}, \tag{2.2.15}$$

where

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h \in O(D,D,\mathbb{Z}) \tag{2.2.16}$$

Then using (2.2.15) and the relation $G' = \frac{1}{2}(E' + E'^t)$, one can show

$$\begin{aligned}(d+cE)^t G'(d+cE) &= G,\\ (d-cE^t)^t G'(d-cE^t) &= G.\end{aligned} \tag{2.2.17}$$

The first relation is obtained by substituting (2.2.15) into E' and the second relation is obtained by inverting (2.2.15) and then substituting the expression in place of E'^t.

The number operators can be calculated using the mode expansions (2.2.13) (2.2.14). They give the commutation relations which the oscillators have to obey. The resulting number operators are given as

$$N = \sum_{k>0}\alpha_{-k}^i(E)G_{ij}\alpha_k^j(E), \quad \bar{N} = \sum_{n>0}\tilde{\alpha}_{-k}^i(E)G_{ij}\tilde{\alpha}_k^j(E) \tag{2.2.18}$$

where the oscillators transform as

$$\begin{aligned}\alpha_k(E) &\to (d-cE^t)^{-1}\alpha_k(E'),\\ \tilde{\alpha}_k(E) &\to (d+cE)^{-1}\tilde{\alpha}_k(E').\end{aligned} \tag{2.2.19}$$

Then together with (2.2.15), it is straightforward that the number operators are invariant under the oscialltor transformations. Thus one can conclude that the normal ordered Hamiltonian (2.2.3) is $O(D, D, \mathbb{Z})$-invariant and hence the physics is equivalent.

3 Double Field Theory

In this section the concept and construction of double field theory are discussed. Also the formulation of T-duality in the context of double field theory is explored.

3.1 Supergravity

To discuss about double field theory, some of the essentials of the massless bosonic sector of string theory need to be addressed. It is worthwhile to remember that this massless bosonic sector is universal in the sense that all closed string theories consisting closed Type I, Type II and heterotic theories mutually share the same action in their full theories. Also one should note that the term 'massless' here applies for the full D-dimensional theory, not massless in the uncompactified theory.

The action for this sector is given as [1]

$$S = \int d^D x \sqrt{g} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H_{ijk} H^{ijk} \right), \tag{3.1.1}$$

where g_{ij} is the metric, R is the Ricci scalar, ϕ is the massless scalar dilaton and $H_{ijk} = 3\partial_{[i}\hat{b}_{jk]}$ is the flux. The $\hat{b}_{ij}$ field is the antisymmetric tensor field.

There are two symmetries which leaves the action invariant. One is diffeomorphism where the fields transform under change of coordinates and the other is the gauge symmetry where the antisymmetric field transforms in accordance to the gauge parameter.

- **Diffeomorphism**: The field transformation due to change of coordinate is most well described using Lie derivatives. Using an arbitrary infinitesimal vector λ^i which parametrises the coordinate change the field transformations can be written as

$$\begin{aligned} g_{ij} &\to g_{ij} + L_\lambda g_{ij} \\ \hat{b}_{ij} &\to \hat{b}_{ij} + L_\lambda \hat{b}_{ij} \\ \phi &\to \phi + L_\lambda \phi, \end{aligned} \tag{3.1.2}$$

where

$$L_\lambda g_{ij} = \lambda^k \partial_k g_{ij} + g_{kj}\partial_i \lambda^k + g_{ik}\partial_j \lambda^k$$
$$L_\lambda \hat{b}_{ij} = \lambda^k \partial_k \hat{b}_{ij} + \hat{b}_{kj}\partial_i \lambda^k + \hat{b}_{ik}\partial_j \lambda^k$$
$$L_\lambda \phi = \lambda^i \partial_i \phi, \tag{3.1.3}$$

and L_λ is the standard Lie derivative.

- **Gauge transformation**: The Kalb-Ramond field is the string theory analogue of electromagnetic field in classical physics. Hence the gauge transformation only affects the $\hat{b}_{ij}$ field in this massless bosonic sector. For some infinitesimal one-form $\tilde{\lambda}_i$, the gauge transformation is formulated as

$$b_{ij} \to b_{ij} + \partial_i \tilde{\lambda}_j - \partial_j \tilde{\lambda}_i \tag{3.1.4}$$

It is useful to divide the fields into two parts, the constant background fields and the fluctuation fields which are affected by the coordinates on the target space. One can also introduce a new scalar field to keep the dilaton, ϕ, invariant under diffeomophisms. Hence the fields can be written as

$$g_{ij} = G_{ij} + h_{ij}$$
$$\hat{b}_{ij} = B_{ij} + b_{ij}$$
$$\phi = d + \tfrac{1}{4} G^{ij} h_{ij}, \tag{3.1.5}$$

where G_{ij} and B_{ij} are background metric and antisymmetric tensor field respectively.

3.2 Linearised Double Diffeomorphism

As brielfly mentioned in the introduction, double field theory introduces an additional set of coordinates often called *dual coordinates* which acts as the counterpart of the spacetime coordinates within the theory. Naturally the three fields involved in the massless sector - gravity field, antisymmetric tensor field and dilaton - depend on two different coodinate systems. Thus the fields should have a double diffeomorpism symmetry where coodinate change of each set of coordinates introduces differomorphism symmetry of its own. To express this mathematically, one can write

$$h_{ij}(x^\mu, x^a, \tilde{x}_a), \hat{b}_{ij}(x^\mu, x^a, \tilde{x}_a), d(x^\mu, x^a, \tilde{x}_a), \tag{3.2.1}$$

where $\mu = 1, ..., d,\ a = 1, ..., n$ for d uncompactified dimensions and n compactified dimensions. The ordinary spacetime coordinate $x^i = (x^\mu, x^a)$ and the dual coordinate $\tilde{x}_i = (0, \tilde{x}_a)$ form the doubled geometry. Note the dual coordinates of the uncompactified dimensions are omitted as they have no physical significance.

A striking feature of double field theory is that in the massless bosonic sector, the existence of a Kalb-Ramond field and a dilaton is required due to the symmetry. On the contrary, conventional string theory does not suggest any firm reason why those fields should be in the same sector. To elaborate, the Einstein's gravity $S = \int \sqrt{g}R$ cannot stay invariant under a double diffeomorphism. This is due to the transformation rules of the h_{ij} field. For simplicity, the background fields will be ignored, focussing only on the fluctuation fields.

Consider the linearised diffeomorphism instead of the full diffeomorphism using the Lie derivatives. Hence the background metric is treated as the flat Minkowski metric. One might propose that the gravity field transforms as

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i, \tag{3.2.2}$$

where $\epsilon_i(x^\mu, x^a, \tilde{x}_a)$ is a gauge parameter. For the dual diffeomorphism, one can also introduce $\tilde{\epsilon}_i(x^\mu, x^a, \tilde{x}_a)$. Thus this leads to the second diffeomorphism

$$\tilde{\delta} h_{ij} = \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i, \tag{3.2.3}$$

To observe how the need of a Kalb-Ramond field and a dilaton arises using these diffeomorphism consider the action only containing the h_{ij} field - Einstein's gravity. Remembering that the metric is of the form (3.1.5) and that $G_{ij} = \eta_{ij}$ where η_{ij} is the Minkowski metric, the action can be written in terms of the h_{ij} field

$$\begin{aligned} S = \int dx d\tilde{x}\ \Bigg(&\frac{1}{4} h^{ij} \partial^2 h_{ij} - \frac{1}{4} h \partial^2 h + \frac{1}{2} (\partial^i h_{ij})^2 + \frac{1}{2} h \partial_i \partial_j h^{ij} \\ &\frac{1}{4} h^{ij} \tilde{\partial}^2 h_{ij} - \frac{1}{4} h \tilde{\partial}^2 h + \frac{1}{2} (\tilde{\partial}^i h_{ij})^2 + \frac{1}{2} h \tilde{\partial}_i \tilde{\partial}_j h^{ij} \Bigg). \end{aligned} \tag{3.2.4}$$

The integration measure $dx d\tilde{x}$ represents integrating over the d-dimensional flat spacetime including the $2n$-dimensional double space. It is straightforward to see that the first line of the action is invariant upon the transformation (3.2.2), and likewise the second line of the action is invariant upon (3.2.3). Hence varying the action with the dual coordinate diffeomorphism gives

$$\tilde{\delta}S = \int dx d\tilde{x} \Big(h^{ij}\partial^2\tilde{\partial}_i\tilde{\epsilon}_j - h^{ij}\partial_i\partial^k\tilde{\partial}_j\tilde{\epsilon}_k + (\partial_i\partial_j h^{ij} - \partial^2 h)\tilde{\partial}\cdot\tilde{\epsilon} + (\partial^i h_{ij} - \partial_j h)(\partial\cdot\tilde{\partial})\tilde{\epsilon}^j \Big). \tag{3.2.5}$$

Then integrating by parts on the first first two terms with relabelling the indices produce

$$\tilde{\delta}S = \int dx d\tilde{x} \Big((\tilde{\partial}_j h^{ij})\partial^k(\partial_i\tilde{\epsilon}_k - \partial_k\tilde{\epsilon}_i) + (\partial_i\partial_j h^{ij} - \partial^2 h)\tilde{\partial}\cdot\tilde{\epsilon} + (\partial^i h_{ij} - \partial_j h)(\partial\cdot\tilde{\partial})\tilde{\epsilon}^j \Big). \tag{3.2.6}$$

Then one can annihilate the first term by introducing a Kalb-Ramond field which varies as $\tilde{\delta}b_{ij} = -(\partial_i\tilde{\epsilon}_j - \partial_j\tilde{\epsilon}_i)$. Likewise introduction of a dilaton varying as $\tilde{\delta}\phi = \frac{1}{2}\tilde{\partial}\cdot\tilde{\epsilon}$ would annihilate the second term of (3.2.6). More precisely addition of the two fields in the action is formulated as

$$S_b = \int dx d\tilde{x} (\tilde{\partial}_j h^{ij})\partial^k b_{ik} \tag{3.2.7}$$

$$S_\phi = -2\int dx d\tilde{x} (\partial_i\partial_j h^{ij} - \partial^2 h)\phi \tag{3.2.8}$$

To deal with the last term, one requires a second=order differential constraint where the gauge parameter needs to be annihilated by $\partial\cdot\tilde{\partial}$.

Thus it has been shown that the introduction of additional fields to the gravity fields are necessary for the formulation of an invariant action under the double diffeomorphism symmetry, and the added fields naturally forms the universal massless bosonic sector of string theory. Moreover, it is emphasised in the original work of [Hull page 9] that the presence of the constraint is also essential as it has not been successful to find a non-trivial theory which is invariant under both of the double diffeomorphism transformations without the aid of the constraint.

3.3 Toroidal Background

Here a more general case of toroidal compactification is discussed. [4] The fundamentals of toroidal compactification begins with the target space string action using sigma-model (2.2.12). The dimensions are categorised as n-dimensional uncompactified and d-dimensional compactified dimensions, hence the coordinates are written as $X^i = (X^a, X^\mu)$ where the $X^a \approx X^a + 2\pi$ are the coordinates for the internal d-dimensional torus.

Unlike in section (2.2) where all dimensions were treated as compactified coordinates, here the flat spacetime are included so the formulation needs to be generalised. The background fields can be expressed as

$$G_{ij} = \begin{pmatrix} \hat{G}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, B_{ij} = \begin{pmatrix} \hat{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.3.1}$$

thus defining E_{ij} field as

$$E_{ij} \equiv G_{ij} + B_{ij} = \begin{pmatrix} \hat{E}_{ab} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}, \hat{E}_{ab} = \hat{G}_{ab} + \hat{B}_{ab} \tag{3.3.2}$$

Here the Hamiltonian is formulated as

$$4\pi H = \big(X', 2\pi P\big)\mathcal{R}(E) \begin{pmatrix} X' \\ 2\pi P \end{pmatrix}, \tag{3.3.3}$$

where $X' = \partial_\sigma X$ and P is the momenta, and $\mathcal{R}(E)$ is the $2D \times 2D$ matrix given in (2.2.4). Using the action and the Hamiltonian one can produce the mode expansions which are given in (2.2.13) for X^i and (2.2.14) for P_i. In double field theory, the mode expansion for the dual coordinates is also required and is given as

$$\begin{aligned} \tilde{X}^i(\tau, \sigma) &= \tilde{x}^i + p_i\sigma + \tau[(G - BG^{-1}B)_{ij}w^j + (BG^{-1})_i^j p_j] \\ &+ \frac{i}{\sqrt{2}} \sum_{l \neq 0} \frac{1}{l}[-E^t_{ij}\alpha^i_l(E)e^{il\sigma)} + E_{ij}\tilde{\alpha}^i_l(E)e^{-il\sigma)}]e^-il\tau. \end{aligned} \tag{3.3.4}$$

With the periodic identification, considering the case where $x^i \approx x^i + 2\pi$ gives integer values for w^a and p_a. Integer w^a thus naturally conjugates to the periodic dual coordinate $\tilde{x}_a \approx \tilde{x}_a + 2\pi$. Remember $w^\mu = 0$ and $\tilde{x}_\mu = 0$, i.e.) the winding modes and dual coordinates in the flat spacetime do not exist.

Now one can add the commutation relation of the dual coordinates to the usual commuatation relation

$$[x^i, p_j] = i\delta^i_j, \quad [\tilde{x}_i, w^j] = i\tilde{\delta}^j_i \tag{3.3.5}$$

where $\tilde{\delta}^i_j = diag(\delta^a_b, 0)$ since dual coordinates are 0 in the Minkowski directions. From (3.3.5) it is possible to obtain the commutation relations for the oscillators which are given as

$$[\alpha^i_m, \alpha^j_n] = [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = mG^{ij}\delta_{m+n,0}. \tag{3.3.6}$$

Then the zero modes take the form

$$\alpha_0^i = \frac{1}{\sqrt{2}} G^{ij}(p_{ij} - E_{jk}w^k)$$
$$\tilde{\alpha}_0^i = \frac{1}{\sqrt{2}} G^{ij}(p_{ij} - E^t_{jk}w^k). \tag{3.3.7}$$

For later use, it is useful to consider the expression where the indices of the zero modes are lowered using the metric.

$$\alpha_{0i} = -i\sqrt{\frac{\alpha'}{2}} D_i, \quad D_i = \frac{i}{\sqrt{\alpha'}}\left(\frac{\partial}{\partial x^i} - E_{ik}\frac{\partial}{\partial \tilde{x}_k}\right) \tag{3.3.8}$$
$$\tilde{\alpha}_{0i} = -i\sqrt{\frac{\alpha'}{2}} \tilde{D}_i, \quad \tilde{D}_i = \frac{i}{\sqrt{\alpha'}}\left(\frac{\partial}{\partial x^i} + E^t_{ik}\frac{\partial}{\partial \tilde{x}_k}\right). \tag{3.3.9}$$

The derivatives $D_i, \tilde{D}_i$ are introduced for convenience later, and note upon lowering the indices $p_j = \frac{1}{i}\partial_j$ and $w^k = \frac{1}{i}\tilde{\partial}^k$ are used. The derivatives constructed are significant in the sense that they are dependent on the left and right-moving coordinates $\tilde{x}_i + E^t_{ij}x^j$ and $\tilde{x}_i - E_{ij}x^j$ respectively. An example of the left and right- moving coordinates were given in section (2.1) where the left-moving X_L coordinate is a fucntion of $(\tau+\sigma)$ and the right-moving coordinate X_R is a function of $(\tau-\sigma)$. This is also the case in this context, as the left-moving $\tilde{X}_i + E^t_{ij}X^j$ is a function of $(\tau+\sigma)$ and the right-moving $\tilde{X}_i - E_{ij}x^j$ is a function of $(\tau-\sigma)$. Again, the independence of the dual coordinates in the Minkowski directions imply that $\frac{\partial}{\partial \tilde{x}_i} = \left(\frac{\partial}{\partial \tilde{x}_a}, 0\right)$ and hence $D_a \neq \tilde{D}_a$ while $D_\mu = \tilde{D}_\mu$.

It is also useful to define two Laplacian operators

$$-\frac{\alpha'}{2}\Box \equiv \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j + \frac{1}{2}\tilde{\alpha}_0^i G_{ij}\tilde{\alpha}_0^j, \quad \Box = \frac{1}{2}(D^2 + \tilde{D}^2), \tag{3.3.10}$$
$$-\frac{\alpha'}{2}\triangle \equiv \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j - \frac{1}{2}\tilde{\alpha}_0^i G_{ij}\tilde{\alpha}_0^j, \quad \triangle = \frac{1}{2}(D^2 - \tilde{D}^2). \tag{3.3.11}$$

To elaborate, the two operators can be rewritten as

$$\Box = \frac{1}{\alpha'}\partial^t \mathcal{R}(E)\partial, \tag{3.3.12}$$
$$\triangle = -\frac{1}{\alpha'}\partial^t \eta \partial, \tag{3.3.13}$$

where

$$\partial = \begin{pmatrix} \frac{\partial}{\partial \tilde{x}_i} \\ \frac{\partial}{\partial x^j} \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3.3.14}$$

Thus $\Box$ is the Laplacian for the generalised metric $\mathcal{R}(E)$ and $\triangle$ is the Laplacian for the $O(D,D,\mathbb{Z})$-invariant metric η. It is worthwhile to note that the level matching condition can be written as [3]

$$L_0 - \bar{L}_0 = N - \bar{N} - \frac{\alpha'}{2}\triangle = 0, \tag{3.3.15}$$

therefore fields satisfying the condition $N = \bar{N}$ has a constraint $\triangle = 0$.

3.4 Quadratic Action

The action proposed in section (3.1) is not a complete action describing double field theory. In fact one can construct action of higher dimensions. The action in the quadratic order is discussed here.

Following from the conformal field theory one can construct a quadratic action of string field theory in the context of double field theory given as

$$S = \int dx d\tilde{x} \left(\frac{1}{4} e_{ij} \Box e^{ij} + \frac{1}{4}(\tilde{D}^j e_{ij})^2 + \frac{1}{4}(D^i e_{ij})^2 - 2dD^i \tilde{D}^j e_{ij} - 4d\Box d \right), \tag{3.4.1}$$

where $e_{ij} = h_{ij} + b_{ij}$ is the fluctuation field and d is the scalar field.

The detailed discussion of how this form can be obtained is given in [3]. For the scope of this dissertation it is useful to note that there are two relevant vectorial gauge parameters which generate the gauge transformations given as

$$\delta_\lambda e_{ij} = \tilde{D}_j \lambda_i, \quad \delta_\lambda d = -\tfrac{1}{4} D \cdot \lambda, \tag{3.4.2}$$

$$\delta_{\tilde{\lambda}} e_{ij} = D_i \tilde{\lambda}_j, \quad \delta_{\tilde{\lambda}} d = -\tfrac{1}{4} \tilde{D} \cdot \tilde{\lambda}, \tag{3.4.3}$$

where $\lambda, \tilde{\lambda}$ are the vectorial gauge parameters. Following from this the variation of the action which includes the gauge transformation driven by both gauge parameters $\delta = \delta_\lambda + \delta_{\tilde{\lambda}}$ is formulated as

$$\delta S = \int dx d\tilde{x} \left(\frac{1}{2} e^{ij} \triangle (\tilde{D}_j \lambda_i - D_i \tilde{\lambda}_j) + 2d\triangle (D \cdot \lambda - \tilde{D} \cdot \tilde{\lambda}) \right). \tag{3.4.4}$$

With the constraint $\triangle = 0$, it is straightforward that the action variation vanishes. Thus the gauge transformations (3.4.2), (3.4.3) leave the action invariant and it proves that the quadratic action (3.4.1) is a consistent theory of fluctuation fields in a toroidal background.

It can be shown that the double field theory action (3.4.1) recovers the conventional massless bosonic action of the form

$$S_c = \int dx \sqrt{g} e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right). \tag{3.4.5}$$

Firstly the conventional action can be rewritten in terms of the fluctuation fields using (3.1.5) and $H_{ijk} = 3\partial_{[i}\hat{b}_{jk]}$, to the quadratic order as

$$\begin{aligned} S_c &\approx \int dx \frac{1}{4} h^{ij}\partial^2 h_{ij} + \frac{1}{2}(\partial^2 h_{ij})^2 - 2d\partial^i\partial^j h_{ij} - 4d\partial^2 d + \frac{1}{4}b^{ij}\partial^2 b_{ij} + \frac{1}{2}(\partial^j b_{ij})^2 \\ &= \int dx \; L(h,b,d;\partial) \end{aligned} \tag{3.4.6}$$

Now to rewrite (3.4.1), it is convenient to first redefine the coordinates to absorb the constant $\sqrt{\alpha'}$. Also remembering that $E_{ij} = G_{ij} + B_{ij}$, one can rewrite the derivatives and the Laplacian operators as

$$D_i = \partial_i - \tilde{\partial}_i - B_{ik}\tilde{\partial}^k, \tag{3.4.7}$$

$$\tilde{D}_i = \partial_i + \tilde{\partial}_i - B_{ik}\tilde{\partial}^k, \tag{3.4.8}$$

$$\Box = \partial^2 + \tilde{\partial}^2 + (B_{ij}\tilde{\partial}^j)^2 - 2B_{ij}\partial^i\tilde{\partial}^j, \tag{3.4.9}$$

$$\triangle = -2\partial_i\tilde{\partial}^i, \tag{3.4.10}$$

where $\tilde{\partial}$ is defined as

$$\tilde{\partial}_i \equiv G_{ik}\frac{\partial}{\partial \tilde{x}^k}. \tag{3.4.11}$$

Note for expression of the dual derivative $\tilde{D}_i$, $E^t_{ij} = G^t_{ij} + B^t_{ij}$ and that $G_{ji} = G_{ij}$, $B_{ji} = -B_{ij}$. For simplicity consider the background where $B_{ij} = 0$. Then one can rewrite the action (3.4.1) as

$$S = \int dx d\tilde{x} \left(L(h,b,d;\partial) + L(h,b,-d;\tilde{\partial}) + (\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij}) - 4d\partial^i\tilde{\partial}^j b_{ij} \right), \tag{3.4.12}$$

where $L(h,b,-d;\tilde{\partial})$ is the Lagrangian of the same form as in (3.4.6) but with $-d$ and the dual derivative $\tilde{\partial}$ instead of the usual scalar and derivatives.

It is interesting to notice that ignoring the dual coordinate $\tilde{x}$ will reproduce the conventional action consisting of the spacetime coordinates, as one might expect. This is because setting $\tilde{x} = 0$ is equivalent to $\tilde{\partial} = 0$ in our action, only leaving the Lagrangian $L(h,b,d;\partial)$. Likewise it is also possible to consider the case when the fields have no x-dependence. It is not surprising to see that all terms in the action vanish except for the Lagrangian $L(h,b,-d;\tilde{\partial})$ this time.

Moreover, this action includes a new coupling that was not observed before. The $(\partial_k h^{ik})(\tilde{\partial}^j b_{ij}) + (\tilde{\partial}^k h_{ik})(\partial_j b^{ij})$ term introduces a coupling between the metric and the

Kalb-Ramond field and the $-4d\partial^i\tilde{\partial}^j b_{ij}$ term introduces a coupling between the Kalb-Ramond field and the dilaton.

As discussed in section (3.2), there are two symmetries to consider about. To begin with, there is diffeomorphism and in the linearised version of the symmetry the conventional action accepts transformation of the form

$$\delta h_{ij} = \partial_i \epsilon_j + \partial_j \epsilon_i, \quad \delta b_{ij} = 0, \quad \delta d = -\frac{1}{2}\partial \cdot \epsilon. \tag{3.4.13}$$

There is also the second symmetry which is the gauge transformation of the antisymmetric tensor,

$$\delta b_{ij} = -\partial_i \tilde{\epsilon}_j + \partial_j \tilde{\epsilon}_i. \tag{3.4.14}$$

The symmetry of double field theory was already discussed in the first part of this section. However for the ease of comparison to the conventional action, it is convenient to define

$$\epsilon_i \equiv \frac{1}{2}(\lambda_i + \tilde{\lambda}_i), \quad \tilde{\epsilon}_i \equiv \frac{1}{2}(\lambda_i - \tilde{\lambda}_i) \tag{3.4.15}$$

Using this one can merge the two symmetries as transformation with respect to the gauge parameters ϵ and another set of symmetries transforming with respect to $\tilde{\epsilon}$. These transformations can be expressed as

$$\begin{aligned} \delta h_{ij} &= \partial_i \epsilon_j + \partial_j \epsilon_i & \tilde{\delta} h_{ij} &= \tilde{\partial}_i \tilde{\epsilon}_j + \tilde{\partial}_j \tilde{\epsilon}_i \\ \delta b_{ij} &= -(\tilde{\partial}_i \epsilon_j - \tilde{\partial}_j \epsilon_i) & \tilde{\delta} b_{ij} &= -(\partial_i \tilde{\epsilon}_j - \partial_j \tilde{\epsilon}_i) \\ \delta d &= -\frac{1}{2}\partial \cdot \epsilon & \tilde{\delta} d &= \frac{1}{2}\tilde{\partial} \cdot \tilde{\epsilon}. \end{aligned} \tag{3.4.16}$$

Notice the first set of transformations reproduces the transformation of fluctuation fields of the conventional theory. The two sets of transformations are related by the exchange of the gauge paramater $\epsilon \leftrightarrow \tilde{\epsilon}$ as well as the exchange of the derivatives $\partial \leftrightarrow \tilde{\partial}$.

A natural question to ask is whether there is a single transformation law which governs double field theory consisting of both of the dual diffeomorphisms stated above. To date, there is no known single transformation law of the sort including the investigation of the cubic action of double field theory [3]. This phenomenon can be represented by the transformation of the dilaton.

The scalar dilaton should be invariant under a symmetry transformation of the action. Therefore the existence of a dilaton which is invariant under both δ and $\tilde{\delta}$ transformations could be an evidence of the unification of double diffeomorphism. Unfortunately this is not the case in this theory.

Consider the dilaton $\phi \equiv d + \frac{1}{4}G^{ij}h_{ij}$ as defined in section (3.1). Upon the transformation with ϵ the dilaton is invariant as

$$\delta\phi = \delta d + \frac{1}{4}G^{ij}\delta h_{ij} = -\frac{1}{2}\partial \cdot \epsilon + 2 \cdot \frac{1}{4}\partial_i \epsilon^i = 0 \tag{3.4.17}$$

Remember G_{ij} is the constant background metric. However this is not the case for the $\tilde{\epsilon}$ transformations:

$$\tilde{\delta}\phi = \tilde{\delta} d + \frac{1}{4}G^{ij}\tilde{\delta} h_{ij} = \frac{1}{2}\tilde{\partial} \cdot \tilde{\epsilon} + 2 \cdot \frac{1}{4}\tilde{\partial}_i \tilde{\epsilon}^i = \tilde{\partial} \cdot \tilde{\epsilon} \neq 0 \tag{3.4.18}$$

Thus ϕ is the invariant dilaton only in the context of the ϵ transformations. One can find a new dilaton for the $\tilde{\epsilon}$ transformations, given as

$$\tilde{\phi} \equiv d - \frac{1}{4}G^{ij}h_{ij} \tag{3.4.19}$$

Indeed, it is easy to verify that $\tilde{\phi}$ is invariant under the $\tilde{\delta}$ variation but not for δ variation.

This existence of two dilatons in one theory which are not equivalently conserved is an evidence that there is no single diffeomorphism that can unify the two dual diffeomorphisms.

4 T-duality in Double Field Theory

4.1 $O(d, d, \mathbb{Z})$ duality

Formulation of T-duality in the D-dimensional toroidal compactification has been discussed in section (2.2). For a more general picture of double field theory it is better to consider the group $O(d, d, \mathbb{Z})$, a subgroup of $O(D, D, \mathbb{Z})$. This is to make sure that the duality transformations do not affect the uncompactified dimensions.

A fundamental building block of $O(D, D, \mathbb{Z})$ is the invariant metric

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{4.1.1}$$

where $h^t J h = J$ for some element $h \in O(D, D, \mathbb{Z})$. A general element of the group $O(D, D, \mathbb{Z})$ can be written as a $2D \times 2D$ matrix in the form

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.1.2}$$

where a, b, c, d are $D \times D$ matrices. Since the group of interest now is $O(d,d,\mathbb{Z})$, the matrices should be rewritten as

$$a = \begin{pmatrix} \hat{a} & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} \hat{b} & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} \hat{c} & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \hat{d} & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.1.3}$$

which makes it straightforward to construct an $O(d,d,\mathbb{Z})$ element from an $O(D,D,\mathbb{Z})$ element, namely

$$\hat{h} = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}, \quad \hat{h} \in O(d,d,\mathbb{Z}) \tag{4.1.4}$$

Recall $E = G + B$ where E, G, B are the background fields written in matrices instead of in the form A_{ij}. In a d-dimensional toroidal compactification the background matrix obtains the form

$$E = \begin{pmatrix} \hat{E} & 0 \\ 0 & \eta \end{pmatrix}, \tag{4.1.5}$$

where η is the Minkowski metric and $\hat{E} = \hat{G} + \hat{B}$. The $O(D,D,\mathbb{Z})$ group action on E was previously addressed in (2.2.15), and therefore the $O(d,d,\mathbb{Z})$ action on $\hat{E}$ is given as

$$\hat{E}' = \hat{h}(\hat{E}) \equiv (\hat{a}\hat{E} + \hat{b})(\hat{c}\hat{E} + \hat{d})^{-1}. \tag{4.1.6}$$

Recall this non-linear group action can be linearised using the generalised background $\mathcal{R}(E)$ such that $\mathcal{R}(E') = h\mathcal{R}(E)h^t$.

Now everything follows that the relations for $O(D,D,\mathbb{Z})$ can be reduced to a $O(d,d,\mathbb{Z})$ compatible relations by redefining the matrices as hatted matrices, representing the avoidance to affect any of the Minkowski dimensions. For example, (2.2.17) becomes

$$\begin{aligned} (\hat{d} + \hat{c}\hat{E})^t \hat{G}' (\hat{d} + \hat{c}\hat{E}) &= \hat{G}, \\ (\hat{d} - \hat{c}\hat{E}^t)^t \hat{G}' (\hat{d} - \hat{c}\hat{E}^t) &= \hat{G}, \end{aligned} \tag{4.1.7}$$

where $d^t + E^t c^t = \begin{pmatrix} \hat{d}^t + \hat{E}^t \hat{c}^t & 0 \\ 0 & 1 \end{pmatrix}$ and similarly for $d^t - Ec^t$. It is convenient to define $d^t - Ec^t \equiv M$ and $d^t + E^t c^t \equiv \tilde{M}$. Another useful relation using (2.2.15) is [4]

$$\begin{aligned} b^t - Ea^t &= -(d^t - Ec^t)E' \\ b^t + E^t a^t &= (d^t + E^t c^t)E'^t \end{aligned} \tag{4.1.8}$$

4.2 $O(d,d,\mathbb{Z})$-invariant Action

Recall the massless bosonic sector of string action consists of only three fields and the metric and the antisymmetric tensor can be merged into a single field e_{ij}. Then the quadratic action (3.4.1) can be expressed with its parameters as

$$S(E,e_{ij},d)=\int dX\ L\left(D_k,\tilde{D}_l,G,e_{ij}(X),d(X)\right), \tag{4.2.1}$$

where $X=\begin{pmatrix}\tilde{x}_i\\ x^i\end{pmatrix}$ is a $2D$ column vector and the insignificant dual coordinate in the Minkowski dimensions is added for mathematical convenience.

Now consider an $O(d,d,\mathbb{Z})$ element h which can also be expressed as an $O(D,D,\mathbb{Z})$ element using the formulations described in section (4.1). Then the $O(D,D,\mathbb{Z})$ action on X is equivalent to diffeomorphism in the doubled torus i.e.) the basis change : $h:X\to X'$, while keeping the Minkowski coordinates conserved. Such action can be expressed as

$$X'=\begin{pmatrix}\tilde{x}'\\ x'\end{pmatrix}=hX=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\tilde{x}\\ x\end{pmatrix}=\begin{pmatrix}a\tilde{x}+bx\\ c\tilde{x}+dx\end{pmatrix}. \tag{4.2.2}$$

If double field theory is compatible with T-duality, the action (4.2.1) must be invariant under the $O(d,d,\mathbb{Z})$ action which is a generalisation of T-duality symmetry.

First the derivatives D and $\tilde{D}$ are treated. Consider a function of X. Then its derivatives must be of the form

$$\begin{aligned}\frac{\partial}{\partial x}F(X')&=\left(b^t\frac{\partial}{\partial\tilde{x}'}+d^t\frac{\partial}{\partial x'}\right)F(X')\\ \frac{\partial}{\partial\tilde{x}}F(X')&=\left(a^t\frac{\partial}{\partial\tilde{x}'}+c^t\frac{\partial}{\partial x'}\right)F(X'),\end{aligned} \tag{4.2.3}$$

since a,b,c,d are independent of X and x' and $\tilde{x}'$ both depend on x and $\tilde{x}$ at the same time. Then using the definition of D_i and $\tilde{D}_i$ in (3.3.8) and (3.3.9), one can express

$$\begin{aligned}DF(X')&=\frac{1}{\sqrt{\alpha'}}\left(b^t\frac{\partial}{\partial\tilde{x}'}+d^t\frac{\partial}{\partial x'}-E(a^t\frac{\partial}{\partial\tilde{x}'}+c^t\frac{\partial}{\partial x'})\right)F(X')\\ &=\frac{1}{\sqrt{\alpha'}}\left((d^t-Ec^t)\frac{\partial}{\partial x'}+(b^t-Ea^t)\frac{\partial}{\partial\tilde{x}'}\right)F(X')\\ &=M\frac{1}{\sqrt{\alpha'}}\left(\frac{\partial}{\partial x'}-E'\frac{\partial}{\partial\tilde{x}'}\right)F(X')\\ &=MD'F(X').\end{aligned} \tag{4.2.4}$$

The third line is obtained using the relation (4.1.7). A short computation of the analogy for $\tilde{D}_i$ gives the identity

$$\tilde{D}F(X') = \tilde{M}\tilde{D}'F(X') \tag{4.2.5}$$

Thus there is an identity for the derivatives, namely

$$D = MD', \quad \tilde{D} = \tilde{M}\tilde{D}' \tag{4.2.6}$$

Now consider the other parameters of the action. The background metric G is manifestly $O(d, d, \mathbb{Z})$-invariant. Also there is an ansatz for $e_{ij}(X)$ and $d(X)$ fields proposed in [Kugo Zwei]

$$e_{ij}(X) = M_i{}^k M_j{}^l e'_{kl}(X') \tag{4.2.7}$$

$$d(X) = d'(X'). \tag{4.2.8}$$

Now the action (4.2.1) can be rewritten in terms of the transformed coordinates under $O(d, d, \mathbb{Z})$ action:

$$S[E, e_{ij}(X(X')), d(X(X')] = \int dX' \; L\left(MD', \; \tilde{M}\tilde{D}', \; G, \; M_i{}^k M_j{}^l e'_{ij}(X'), \; d'(X')\right), \tag{4.2.9}$$

where the integration measure $dX = dX'$ since

$$\int dX = \int dx^\mu dx^a d\tilde{x}_a = \int dx^\mu dx'^a d\tilde{x}'_a = \int dX', \tag{4.2.10}$$

contracted indices are not affected by coordinate transformation.

Duality invariance is achieved if the rewritten action (4.2.9) equals to the action which undergoes explicit coordinate change. i.e.) if $S[E, e_{ij}(X(X')), d(X(X')] = S[E', e'_{ij}(X'), d'(X')]$, where

$$S[E, e_{ij}(X'), d(X')] = \int dX' \; L\left(D', \; \tilde{D}', \; G, \; e'_{ij}(X'), \; d'(X')\right), \tag{4.2.11}$$

then the action is preserved through $O(d, d, \mathbb{Z})$ duality action and hence one can state the theory is duality invariant.

To see this, one needs to focus carefully on how the indices contract within the action. Two sets of indices can be assigned to the parameters, the unbarred indices i which transform according to the M matrix and the barred indices which transform according to the $\tilde{M}$ matrix. This is because the indices involved with M is never contracted by the indices involved with $\tilde{M}$ and vice-versa. It follows that all the terms in the action can be assigned with either barred or unbarred indices without the mixing of the two kinds. For example in (4.1.7) it is demonstrated that the metric G^{ij} has either both indices unbarred or both indices barred, but never have one unbarred and the other barred. The only exception is the e_{ij} field, which has both indices at the same time - the first index is an unbarred index and the second index is a barred index. Hence writing out the indices explicitly, the parameters in the action can be written as

$$D_i, \quad \tilde{D}_{\bar{j}}, \quad G^{ij} or\ G^{\bar{i}\bar{j}}, \quad e_{i\bar{j}}, \quad d$$

Notice how all the indices are lowered except for the metric, and the metric transforms as given in (4.1.7). In matrix format this can be expressed as

$$G^{-1} = (\tilde{M}^t)^{-1} G'^{-1} \tilde{M}^{-1}, \tag{4.2.12}$$

$$G^{-1} = (M^t)^{-1} G'^{-1} M^{-1}. \tag{4.2.13}$$

Thus that when one examines how a term with contracted indices transform under the coordinate change transformation law, because of the behaviour the metric shows in such transformations all the M and $\tilde{M}$ matrices naturally annihilate themselves. Thus the statement

$$S[E, e_{ij}(X(X')), d(X(X')] = S[E', e'_{ij}(X'), d'(X')] \tag{4.2.14}$$

is proved and this represents that double field theory incorporates T-duality.

5 Dimensional Reduction

Because the physics one can explicitly observe only consists of the four dimensional world, there is a need to reduce the dimension of the full string theory to four dimensional. This is also the case for double field theory.

5.1 Scherk-Schwarz(SS) Compactification

The massless bosonic sector of supergravity consisting the metric, the antisymmetric tensor and a dilaton will be treated in this section. A detailed analysis of the SS compactification is given in [18,19], as well as a simple review of the procedure in [5]. First the coordinates need to be splitted. There are d-dimensional compactified coordinates and n-dimensional uncompactified coordinates. It is simple to denote this as

$$x^i = (x^\mu, y^a) \tag{5.1.1}$$

where $\mu = 1, ..., n$ and $a = 1, ..., d$.
In toroidal compactification the next step was to impose the periodic identification, but for general SS compactification this is not the case. Also now consider a more general scheme where the uncompactified dimensions are not Minkowski, but rather a general metric g.

In a low dimensional theory, there would be a mixed component of the fields where the fields from a the original theory affects the effective theory. Hence the fields can be splitted into block matrices

$$g_{ij} = \begin{pmatrix} g_{\mu\nu} + g_{pq}A^p{}_\mu A^q{}_\nu & A^p{}_\mu g_{pn} \\ g_{mp}A^p{}_\nu & g_{mn} \end{pmatrix}, \tag{5.1.2}$$

$$b_{ij} = \begin{pmatrix} b_{\mu\nu} - \frac{1}{2}(A^p{}_\mu V_{p\nu} - A^p{}_\nu V_{p\mu}) + A^p{}_\mu A^q{}_\nu b_{pq} & V_{n\mu} - b_{np}A^p{}_\mu \\ -V_{m\nu} + b_{mp}A^p{}_\nu & b_{mn} \end{pmatrix}, \tag{5.1.3}$$

where $A^p{}_\mu$ and $V_{p\mu}$ are vectors in the uncompactified dimensions, but affected by the (0,2)-tensors in the compactified dimensions. Each blocks are constructed in such a way to conserve the symmetric/antisymmetric properties of the fields in the most general layout. In addition, the gauge parameters also split as

$$\lambda^i = (\epsilon^\mu, \Lambda^m), \quad \tilde{\lambda}^i = (\epsilon_\mu, \Lambda_m). \tag{5.1.4}$$

Now one proposes an ansatz for the components. In the Scherk-Schwarz compactification, the fields' dependence to the compactified and uncompactified coordinates take the form

$$\begin{aligned} g_{\mu\nu} &= \hat{g}_{\mu\nu}(x) & b_{\mu\nu} &= \hat{b}_{\mu\nu}(x), \\ A^m_\mu &= u_a{}^m(y)\hat{A}^a{}_\mu(x), & V_{m\mu} &= u^a{}_m(y)\hat{V}_{a\mu}(x), \\ g_{mn} &= u^a{}_m(y)u^b{}_n(y)\hat{g}_{mn}(x), & b_{mn} &= u^a{}_m(y)u^b{}_n(y)\hat{b}_{mn}(x), \end{aligned}$$
$$\phi = \hat{\phi}(x), \tag{5.1.5}$$

where the function which carries the dependence of the compactified dimensions is called a twist. The immediate impression is that when the reduction procedure is finished, the y-dependence of the twist will remain in the uncompactified dimensions as a constant that

does not affect the fields of the effective theory. This is a straightforward result, since if it keeps holding the y-dependence the overall Lorentz invariance will be lost in the theory. In the virture of twists, the hatted fields can only depend on the uncompactified coordinates hence surviving the truncation of the higher dimensions.
The gauge parameters are twisted in the same fashion:

$$\lambda^i = \left(\hat{\epsilon}^\mu, u_a{}^m(y)\hat{\Lambda}^a(x)\right), \quad \tilde{\lambda}^i = \left(\hat{\epsilon}_\mu, u_m^{\ a}(y)\hat{\Lambda}_a(x)\right). \tag{5.1.6}$$

The next step is to look for residual gauge transformations that survive in the low dimension. The gauge transformations in the full theory are formulated using the Lie derivatives (3.1.2) which take the form

$$L_\lambda V^i = \lambda^j \partial_j V^i - V^j \partial_j \lambda^i, \tag{5.1.7}$$

for some vector V^i and the gauge parameter λ^i. The residual gauge can be found substituting in the splitted form for the vector, i.e.) $V^i = (\hat{v}^\mu(x), u_a{}^m(y)\hat{v}^a(x))$. Then the resulting Lie derivatives for the two different parts of the vector are

$$L_\lambda V^\mu = \hat{\epsilon}^\nu \partial_\nu \hat{v}^\mu - \hat{v}^\nu \partial_\nu \hat{\epsilon}^\mu = \hat{L}_{\hat{\epsilon}} \hat{v}^\mu, \tag{5.1.8}$$

$$L_\lambda V^m = u_a{}^m \hat{L}_{\hat{\lambda}} \hat{v}^a. \tag{5.1.9}$$

An interesting feature of this procedure is that to covariantise the Lie derivative, it takes the form

$$\hat{L}_{\hat{\lambda}} \hat{v}^a = L_{\hat{\lambda}} \hat{v}^a + f^a_{\ bc} \hat{\Lambda}^b \hat{v}^c, \tag{5.1.10}$$

where the $f^a_{\ bc}$ is called the flux and it is formulated as

$$f^c_{\ ab} = u_a{}^m \partial_m u_b{}^n u^c{}_n - u_b{}^m \partial_m u_a{}^n u^c{}_n. \tag{5.1.11}$$

Becuase the twist depends on the compactified dimensions, it seems odd that such a term should be added to the residual transformations and introduces a y-dependence in the theory. Hence a natural assumption to take is that the flux is a constant that corresponds to the flux of the metric.

To summarise the gauge transformations, it is useful to put the gauge parameters as

$$\hat{\xi} = (\hat{\epsilon}_\mu, \hat{\epsilon}^\mu, \hat{\Lambda}^A), \quad \hat{\Lambda}^A = (\hat{\lambda}_a, \hat{\lambda}^a). \tag{5.1.12}$$

and the vector fields as

$$\hat{A}^A_{\ \mu} = (\hat{V}_{a\mu}, \hat{A}^a{}_\mu) \tag{5.1.13}$$

and the scalars as

$$\hat{M}_{AB} = \begin{pmatrix} \hat{g}^{ab} & -\hat{g}^{ac}\hat{b}_{cb} \\ \hat{b}_{ac}\hat{g}^{cb} & \hat{g}_{ab} - \hat{b}_{ac}\hat{g}^{cd}\hat{b}_{db} \end{pmatrix} \tag{5.1.14}$$

Notice this matrix is in the exact same form as the generalised metric $\mathcal{R}(E)$ for $O(D, D, \mathbb{Z})$. With all the above sorted, all the gauge transformations can be expressed as

$$\delta_{\hat{\xi}}\hat{g}_{\mu\nu} = L_{\hat{\epsilon}}\hat{g}_{\mu\nu}, \tag{5.1.15}$$

$$\delta_{\hat{\xi}}\hat{b}_{\mu\nu} = L_{\hat{\epsilon}}\hat{b}_{\mu\nu} + (\partial_\mu\hat{\epsilon_\nu} - \partial_\nu\hat{\epsilon_\mu}), \tag{5.1.16}$$

$$\delta_{\hat{\xi}}\hat{A}^A{}_\mu = L_{\hat{\epsilon}}\hat{A}^A{}_\mu - \partial_\mu\hat{\Lambda}^A + f_{BC}^{\ \ A}\hat{\Lambda}^B\hat{A}^C{}_\mu, \tag{5.1.17}$$

$$\delta_{\hat{\xi}}\hat{M}_{AB} = L_{\hat{\epsilon}}\hat{M}_{AB} + f_{AC}^{\ \ D}\hat{\Lambda}^C\hat{M}_{DB} + f_{BC}^{\ \ D}\hat{\Lambda}^C\hat{M}_{AB}. \tag{5.1.18}$$

The transformations of the fields with respect to the gauge parameters readily shows which symmetries each component of $\hat{\xi}$ generate. $\hat{\epsilon}^\mu$ generates diffeomorphism, $\hat{\epsilon}_\mu$ generates the gauge transformation of the antisymmetric tensor and $\hat{\Lambda}^A$ generates the gauge transformations of vectors.

Now all the transformations and the ansatz are identified, one can simply substitute the ansatz into the supergravity action to obtain the effective field theory produced by the Scherk-Schwarz compactification.

5.2 Generalised Scherk-Schwarz Compactification

Since double field theory has twice the number of coordinates compared to the conventional theory, to use the Scherk-Schwarz compactification to double field theory one needs to generalise the procedure into $2D$-dimensional one. Such method was proposed by [20,21].

To mention some of the first steps of the procedure, first split the coordinates as in the original Scherk-Schwarz compactification. Available coordinates are $x^i = (x^m u, y^m)$, $\tilde{x}_i = (\tilde{x}_\mu,\ \tilde{y}_m)$. Therefore split them into the uncompactified dimensions $X = (x^\mu,\ \tilde{x}_\mu)$ and the compactified dimensions $Y = (y^m, \tilde{y}_m)$.

Then introduce the ansatz for the fields and the gauge parameters.

$$E^{\bar{A}}{}_M(X) = \hat{E}^{\bar{A}}_I(X)U^I{}_M(Y), \tag{5.2.1}$$

$$d(X) = \hat{d}(X) + \lambda(Y), \tag{5.2.2}$$

$$\xi^M(X) = \hat{\xi}^I(X)U_I{}^M(Y), \tag{5.2.3}$$

where the indices span $1, ..., 2D$. Again there is the twist which absorbs all the Y-dependence in the theory. In doing so, because the twist is a field in the compactified theory it must obey the double field theory symmetry, namely the $O(d, d, \mathbb{Z})$ symmetry. Hence the twist is an element of $O(d, d, \mathbb{Z})$.

An advantage of compactifying a double field theory action compared to the conventional action is that double field theory only deals with two fields, the combined metric consisting both the metric and the antisymmetric tensor and the scalar field. This leads to the unification of transformation laws treated in the conventional case (5.1.15)-(5.1.18). However to do this the Lie derivatives must be generalised as well as the fluxes, and a detailed analysis of such procedure is given in [22]

There is a proposal that after the generalised Scherk-Schwarz compactification, the double field theory becomes gauged producing the gauged double field theory [5]. An interesting feature of gauged double field theory is that just like the original double field theory is an extension of supergravity with imposing the $O(D, D, \mathbb{Z})$ symmetry, the gauged double field theory is related to the gauged supergravity (the reduced supergravity in low dimensions) by the $O(d, d, \mathbb{Z})$ subgroup of $O(D, D, \mathbb{Z})$. Nonetheless, there are subtle differences between the gauged double field theory and the gauged supergravity, and hence this relation could lead to a new insight in the geometrical structure of string theory.

6 Conclusion

The structure of double field theory, a theory to enhance what is known about string theory, is investigated. It originates from a unique string theory symmetry called T-duality, which involves the momentum and winding modes in closed strings. To formulate closed strings it is essential to impose periodic conditions to the worldsheet in order to create the toroidal environments, however it seems that this ends up in a double torus. Double field theory is an attempt to analyse the fields that live on the double torus, and it has been providing some useful insights such as unification of symmetric and antisymmetric tensors in supergravity.

In this dissertation the method of how T-duality is incorporates in double field theory was focussed mainly, however the general symmetries such as double differomorphism and gauge transformations were also discussed. A natural outlook would include non-linear diffeomorphisms, the nature of massive fields and fermionic fields on the double torus.

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